Classical Hodge theory

X cpt ori. C^∞-man., g riemannian metric on $X \Rightarrow$
$\Rightarrow g$ gives an isomorphism $T_p(X) \to T_p(X)^* \Rightarrow$
$\Rightarrow g$ induces a metric on $\Lambda^* T_p^*(X)$,
$g(u_1\wedge\ldots\wedge u_r, v_1\wedge\ldots\wedge v_r) = \det(g(u_i, v_j))$.

Hodge $*$ operator Let $e_1,\ldots,e_N$ be an orthonormal frame for $T_p^*(X)$ which is positively ori..

$J = \{j_1,\ldots,j_r\}$, $e_J = e_{j_1}\wedge\ldots\wedge e_{j_r}$, $K = \{1,\ldots,N\}\setminus J$,
$e_J^* = \pm \bigwedge_{\ell\in K} e_\ell$ with sign s.t. $e_J \wedge e_J^* = e_1\wedge\ldots\wedge e_N$.
$\langle \alpha,\beta\rangle = \int_X \alpha\wedge \underset{\text{defined by linear extension}}{*\beta}$ is an inner product on $\Lambda^*(X)$.

$\langle g\cdot u, g\cdot v\rangle = \langle u, v\rangle$, $g\in SO_N$.

G finite group, $\rho: G\to GL(V)$. If $(-,-)$ is an inner product on V, $\langle\alpha,\beta\rangle = \frac{1}{|G|}\sum(g\cdot\alpha, g\cdot\beta)$ is a G-invariant " ".
$(-,-)$ inner product on $V \Rightarrow$ the corresponding special orthogonal group G is cpt, $\langle\alpha,\beta\rangle = \frac{1}{\mathrm{Vol}(G)}\int_G (g\cdot\alpha, g\cdot\beta)\,dV$ is a G-inv. inner product.
Haar measure

Ex.: $\mathbb{C}\cong\mathbb{R}^2$. $\partial/\partial y$, $\partial/\partial x$ Asume $\{dx, dy\}$ is positive ori..
$\langle\partial/\partial x, \partial/\partial x\rangle = \langle\partial/\partial y,\partial/\partial y\rangle = 1$,
$\langle\partial/\partial x,\partial/\partial y\rangle = 0$.

$*dx = dy$, $*dy = -dx$.
$f(x,y)$ harmonic function: $\Delta f = 0$, $\Delta = \overset{\text{because reasons}}{-}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$.
$dz = dx + i\,dy$, $d\bar z = dx - i\,dy$,
$\frac{\partial}{\partial z} = \frac12\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar z} = \frac12\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.
$\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} = \frac14\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \Rightarrow \Delta = -4\frac{\partial}{\partial z}\frac{\partial}{\partial\bar z}$.

Ex.: $f(x,y) = x$ is harmonic, $df = dx$,
$df + i*df = dx + i\,dy = dz$ hol. 1-form.
In general: $df + i*df = f_x dx + f_y dy + i f_x dy - i f_y dx =$
$= (f_x - i f_y)dx + (f_y + i f_x)dy =$
$= (f_x - i f_y)dx + i(f_x - i f_y)dy = (f_x - i f_y)(dx + i\,dy) =$
$= (f_x - i f_y)dz = 2\frac{\partial f}{\partial z}dz$. It is a hol. 1-form. $\Leftrightarrow \frac{\partial}{\partial\bar z}\left(\frac{\partial f}{\partial z}\right) = 0 \Leftrightarrow$
$\Leftrightarrow f$ is harmonic.

Lemma: $\dim V = N \Leftrightarrow **\alpha = (-1)^{k(N-k)}\alpha$, $\alpha\in\Lambda^k(V)$.

Cor.: $g(*\alpha,*\beta) \underset{\alpha,\beta\in\Lambda^k(X)}{=} \int_X (*\alpha)\wedge*(*\beta) = \int_X *\alpha\wedge(-1)^{k(N-k)}\beta =$
$= \int_X \beta\wedge*\alpha = g(\beta,\alpha) = g(\alpha,\beta)$.

$\alpha\in\Lambda^k(X)$, $\beta\in\Lambda^{k+1}(X)$, $g(d\alpha,\beta) = \int_X d\alpha\wedge*\beta = g(\alpha,\delta\beta)$,
δ adjoint of d. We want a formula for δ.
$\delta\beta = \mu * d * \beta$.
$\int_X d\alpha\wedge*\beta \underset{\substack{\text{Stokes,}\\ \partial X=\emptyset\\ (X\text{ cpt})}}{=} -\int_X \alpha\wedge d*\beta = -\int_X (-1)^{k(N-k)}\alpha\wedge*(*d*\beta) =$
$= -\frac{(-1)^{k(N-k)}}{\mu}\int_X \alpha\wedge*(\delta\beta) = -\frac{(-1)^{k(N-k)}}{\mu}g(\alpha,\delta\beta) \Rightarrow$
$\Rightarrow \mu = (-1)^{k(N-k)+1} = (-1)^{k(N-1)+1}$.

$\Delta := d\delta + \delta d$.
$g(\Delta\alpha,\alpha) = g(d\delta\alpha,\alpha) + g(\delta d\alpha,\alpha) = g(\delta\alpha,\delta\alpha) + g(d\alpha,d\alpha) \geq 0$.
If $\Delta\alpha = \lambda\alpha$ then $g(\Delta\alpha,\alpha) = \lambda g(\alpha,\alpha) \geq 0$ ($g(\alpha,\alpha)>0 \Rightarrow \lambda\geq 0$).
Suppose $\Delta\alpha = 0 \Rightarrow g(\delta\alpha,\delta\alpha) = g(d\alpha,d\alpha) = 0 \Rightarrow d\alpha = 0$, $\delta\alpha = 0$.
$\Delta\alpha = 0 \Rightarrow \alpha$ is closed.

Thm. (Hodge decomposition): let X be a cpt, ori. riemannian man.. Then:
1) $\mathrm{Harm}^k(X) = \{\alpha\in\Lambda^k(X) \mid \Delta\alpha = 0\}$ is finite dim.;
2) $\Lambda^k(X) = \mathrm{Harm}^k(X)\oplus d\Lambda^{k-1}(X)\oplus\delta\Lambda^{k+1}(X)$
(orthogonal decomposition).

Cor.: every de Rham cohomology class has a unique harmonic representative.

Proof: let α be a representative of $[\alpha]$.
$\Lambda^k(X)\ni\alpha = \alpha_h + d\alpha' + \delta\alpha''$, $\Delta\alpha_h = 0$.
$0 = d\alpha = d\alpha_h + d^2\alpha' + d\delta\alpha'' = d\delta\alpha'' \Rightarrow g(d\delta\alpha'',\alpha'') = 0 \Rightarrow$
$\Rightarrow g(\delta\alpha'',\delta\alpha'') = 0 \Rightarrow \delta\alpha'' = 0 \Rightarrow [\alpha] = [\alpha_h]$.
$\tilde\alpha_h$ another harm. repr. $\Rightarrow \tilde\alpha_h = \alpha_h + d\beta$. $\alpha_h, \tilde\alpha_h$ harm. $\Rightarrow$
$\underset{0}{\delta\alpha_h} = \underset{0}{\delta\tilde\alpha_h} + \delta d\beta \Rightarrow \delta d\beta = 0 \Rightarrow g(\delta d\beta,\beta) = 0 \Rightarrow$
$\Rightarrow g(d\beta, d\beta) = 0 \Rightarrow d\beta = 0$. $\square$