Kähler metrics

X comp. man., J comp. struct., g riemannian metric on X.
g is hermitian if $g(J\alpha, J\beta) = g(\alpha, \beta)$.
Associated 2-form $\omega(u,v) = g(Ju, v)$. $J^2 = -1 \Rightarrow$
$\Rightarrow \omega(u,v) = -\omega(v,u)$. Associated hermitian metric:
$h = g - i\omega$, $d = \mathrm{Re}\, h$, $\omega = -\mathrm{Im}\, h$.
$\omega(Ju, Jv) = g(J^2u, Jv) = -g(u, Jv) = -g(Ju, J^2 v) =$
$= g(Ju, v) = \omega(u,v)$.

Kähler condition: g is Kähler if $d\omega = 0$.

Remark: $g \rightsquigarrow \nabla$ (LC connection). For Kähler $\exists J$.
$d\omega = 0 \Leftrightarrow \nabla J = 0$.

Prop.: if (X, h) is Kähler and Y is a comp. subman. of X, then $(Y, h_{|Y})$ is Kähler.

Proof: $i: Y \to X$ inclusion $\Rightarrow d i^* \omega_X = i^* d\omega_X = 0$ and
$i^*\omega_X = \omega_Y$. $\square$

Ex.: standard metric on $\mathbb{C}^N$ is Kähler.
$$h = \sum_{j=1}^N dz_j \otimes d\bar{z}_j = \sum (dx_j \otimes dx_j + dy_j \otimes dy_j) + \underbrace{-\frac{1}{2} \sum dz_j \wedge d\bar{z}_j}_{\text{closed}}.$$

Remark: h Kähler $\Leftrightarrow$ h is "euclidean" up to scalar order in correct coord. system.

Ex.: $\mathbb{C}^N / \Lambda$, $\mathrm{rk}\, \Lambda = 2N$.

Ex.: Fubini-Study metric on $\mathbb{P}^N$:
$\omega = \frac{i}{2} \partial\bar{\partial} \log |z|^2$, $z = [z_0 : \dots : z_N]$.
f holo. $\Rightarrow \log |f|^2$ harmonic $\Rightarrow \omega$ is well def.
$\log |f|^2 = \log f\bar{f} = \log f + \log \bar{f} + \text{constant} \Rightarrow$
$\partial\bar{\partial} \log |f|^2 = \partial\bar{\partial}(\log f + \log \bar{f}) = \partial\bar{\partial} \log f - \bar{\partial}\partial \log \bar{f} = 0 - 0 = 0$.

Ex.: X cpt Kähler $\Rightarrow$ all odd Betti numbers are even.

Hodge $*$: conj. linear map $\bar{*}: \mathcal{E}^{p,q}(X) \to \mathcal{E}^{N-p, N-q}(X)$,
$\bar{*}\alpha = * \bar{\alpha}$, $h(\alpha, \beta) = \int_X \alpha \wedge \bar{*}\beta$.

Complex laplacian

$d = \partial + \bar{\partial}$, $\bar{\partial}^*$ adjoint of $\bar{\partial}$, $\Delta_d = dd^* + d^*d$ $(d^* = \delta)$,
$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, $\Delta_\partial = \partial\partial^* + \partial^*\partial$, ∂^* adjoint of ∂
(the adjoints are taken w.r.t. h).

Thm.: X cpt Kähler $\Rightarrow \Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_\partial$.

Thm.: $\mathrm{Harm}_d^k(X) = \bigoplus_{p+q=k} \mathrm{Harm}_{\bar{\partial}}^{p,q}(X)$.

Proof (of second thm.): 1) $\sum \mathrm{Harm}_{\bar{\partial}}^{p,q}(X)$ is direct because $\mathcal{E}^k(X) = \bigoplus \mathcal{E}^{p,q}(X)$.
2) $\Delta_d = 2\Delta_{\bar{\partial}}$, so $\Delta_{\bar{\partial}}\alpha = 0 \Rightarrow \Delta_d \alpha = 0$, hence $\supseteq$ ok.
3) Suppose $\Delta_d \alpha = 0$, $\alpha = \sum_{p+q=k} \alpha^{p,q}$, $\alpha^{p,q} \in \mathcal{E}^{p,q}(X)$.
$\Delta_d \alpha = 0 \Rightarrow \Delta_{\bar{\partial}} \alpha = 0 \Rightarrow (\Delta_{\bar{\partial}}\alpha)^{p,q} = 0 \underset{X \text{ Kähler}}{\Rightarrow} \Delta_{\bar{\partial}} \alpha^{p,q} = 0$. $\square$

Note: $\overline{\mathrm{Harm}_{\bar{\partial}}^{p,q}(X)} = \mathrm{Harm}_{\bar{\partial}}^{q,p}(X)$.

Cor.: if X is cpt Kähler, than the odd Betti numbers of X are even.

Proof: $H^k(X, \mathbb{C}) \cong \mathrm{Harm}_d^k(X) \cong \bigoplus_{p+q=k} \mathrm{Harm}_{\bar{\partial}}^{p,q}(X)$.
If k is odd, use the note. $\square$