$$\overline{\mathrm{Harm}_{\bar\partial}^{p,q}(X)} \cong \mathrm{Harm}_{\bar\partial}^{q,p}(X)$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ H^{p,q}(X) & & H^{q,p}(X) \\ \cap & & \cap \\ H^{k}_{DR}(X;\mathbb{C}) & & H^{k}_{DR}(X;\mathbb{C}) \end{array} \quad \rightsquigarrow \quad H^{p,q}(X) \cong H^{q,p}(X)$$

$\mathbb{C}_X$ constant sheaf on X,

$\mathbb{C}_X \xhookrightarrow{i} \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \dots, \; d^2=0$

$$H^k(X;\mathbb{C}) = \frac{\mathrm{Ker}(d:\mathcal{E}^k_X \to \mathcal{E}^{k+1}_X)}{\mathrm{Im}(d:\mathcal{E}^{k-1}_X \to \mathcal{E}^k_X)} \quad \text{acyclic resolution}$$

Dolbeault cohomology: $\Omega^p \hookrightarrow \mathcal{E}^{p,0} \xrightarrow{\bar\partial} \mathcal{E}^{p,1} \xrightarrow{\bar\partial} \mathcal{E}^{p,2} \xrightarrow{\bar\partial} \dots,$

$$H^q(X,\Omega^p) \cong \frac{\mathrm{Ker}(\bar\partial:\mathcal{E}^{p,q}_X \to \mathcal{E}^{p,q+1}_X)}{\mathrm{Im}(\bar\partial:\mathcal{E}^{p,q-1}_X \to \mathcal{E}^{p,q}_X)}.$$

Claim: $\underbrace{H^q(X,\Omega^p)}_{\text{depends only on the complex structure of } X} \cong \mathrm{Harm}_{\bar\partial}^{p,q}(X)$.

We know that if $X=\mathbb{C}/\Lambda$ elliptic curve then $\mathrm{Harm}_{\bar\partial}^{1,0} = \underbrace{H^0(X,\Omega^1)}_{\text{holo. 1-forms on } X}$.

Now we prove the claim.

$\Lambda^{p,q}(X) \cong \mathrm{Harm}_{\bar\partial}^{p,q}(X) \oplus \bar\partial\Lambda^{p,q-1}(X) \oplus \bar\partial^*\Lambda^{p,q+1}(X)$, ← the sum is direct

$$H_{\bar\partial}^{p,q}(X) = \frac{\mathrm{Ker}(\bar\partial:\Lambda^{p,q}(X) \to \bar\partial^{p,q+1}(X))}{\mathrm{Im}(\bar\partial:\Lambda^{p,q-1}(X) \to \Lambda^{p,q}(X))}.$$

$\alpha \in \mathrm{Harm}_{\bar\partial}^{p,q}(X) \Rightarrow \langle (\bar\partial\bar\partial^*+\bar\partial^*\bar\partial)\alpha,\alpha\rangle = 0 \Rightarrow \bar\partial\alpha=0,$

$\mathrm{Harm}_{\bar\partial}^{p,q}(X) \to H_{\bar\partial}^{p,q}(X)$; is it inj.?

$\alpha \longmapsto [\alpha]$

$[\alpha]=0 \Rightarrow \alpha=\bar\partial\beta,\ \beta\in\Lambda^{p,q-1}(X)$, but $\mathrm{Harm}_{\bar\partial}^{p,q}(X) \cap \bar\partial\Lambda^{p,q-1}(X) = \varnothing$

so $\alpha=0$.

Is it surj.? $\alpha\in\Lambda^{p,q}(X),\ \bar\partial\alpha=0,\ \alpha = \underset{\mathrm{Harm}_{\bar\partial}^{p,q}(X)}{\alpha_h} + \bar\partial \underset{\Lambda^{p,q-1}(X)}{\alpha'} + \bar\partial^* \underset{\Lambda^{p,q+1}(X)}{\alpha''},$

$0=\bar\partial\alpha \Rightarrow \bar\partial\bar\partial^*\alpha''=0 \Rightarrow \bar\partial^*\alpha''=0 \Rightarrow [\alpha]=[\alpha_h]$.

The claim follows.

Cor.: let $f:X\to Y$ be a holo. map between cpt Kähler man..

Then $f^*: H^k_{DR}(Y;\mathbb{C}) \to H^k_{DR}(X;\mathbb{C})$ maps $H^{p,q}(Y)$ to $H^{p,q}(X)$.

Proof: $f^*\bar\partial = \bar\partial f^*$. $\square$

Def.: an abstract Hodge structure of weight k consists of a finite rank $\mathbb{Z}$-module $H_\mathbb{Z}$ together with a decomposition $H_\mathbb{C} = H_\mathbb{Z}\otimes\mathbb{C} = \bigoplus_{p,q} H^{p,q}$, $\overline{H^{p,q}} = H^{q,p}$.

Note: the data $\{H^{p,q}\}$ is equivalent to $F^pH_\mathbb{C} = \bigoplus_{a\geq p} H^{a,k-a}$ by $p+q=k \Rightarrow H^{p,q} = F^pH_\mathbb{C} \cap \overline{F^qH_\mathbb{C}}$. It's a check.

To go the other way, we need $F^pH_\mathbb{C} \oplus \overline{F^{k-p+1}H_\mathbb{C}} = H_\mathbb{C}$ and $F^\bullet H_\mathbb{C} \rightsquigarrow 0 \subseteq \dots \subseteq H_\mathbb{C}$ $(F^pH_\mathbb{C} \subseteq F^{p-1}H_\mathbb{C})$.

It's another check.