Conic bundles: X cubic 3-fold, $H^{3,0}(X)=0$, $f: X \dashrightarrow \mathbb{P}^2$ with conics as fibers. Pick $\ell \subseteq X$ line, $X \subseteq \mathbb{P}^3 \rightsquigarrow \ell = V(\ell_1, \ell_2, \ell_3)$, $\hookrightarrow$ linear

$f(x) = [\ell_1(x) : \ell_2(x) : \ell_3(x)] \rightsquigarrow$ fibers of f are conics of dim. 1 ($\dim X = \dim(\text{base}) + \dim(\text{fiber})$).

Conics of dim. 1 in $\mathbb{P}^2$: $\bigcirc$, $\times$, $/\!/$.

$\rightsquigarrow$ we have a branch curve $C \subseteq \mathbb{P}^2$, $\tilde{C} \xrightarrow{\pi} C$ 2:1.

$\chi(\tilde{C}) = 2\chi(C)$, $\deg C = 5 \Rightarrow g(C) = \frac{(5-1)(5-2)}{2} = 6 \Rightarrow$

$\Rightarrow \dim P(C) = 5$.

$\#$pol. of deg. 3 in $\mathbb{P}^4 = 35 \rightsquigarrow$

$\rightsquigarrow \underbrace{(35-1)}_{\text{parameters}} - (\dim GL_5 - 1) = 10$.

Fano surfaces

X smooth cubic 3-fold in $\mathbb{P}^4$, $S = \{\ell \in \mathbb{P}^4 \mid \ell \subseteq X\}$ surface in X, $S \hookrightarrow Gr(2,5) \xrightarrow{\text{Plucker}} \mathbb{P}^9$. Hodge diamond of S:

$$\begin{array}{ccccc} & & 1 & & \\ & 5 & & 5 & \\ 10 & & 25 & & 10 \\ & 5 & & 5 & \\ & & 1 & & \end{array}$$

$10 \to \mathrm{Sym}^2$ or Λ^2, but Λ^2 would be $-1 \Rightarrow$ $\Rightarrow$ it's Sym^2

$5 \quad 5 \to P(C)$

Thm. (tangent bundle [Fano, Griffiths-Clemens]): let U be the restriction of the universal rank 2 bundle on $G(2,5)$ to S; then $U \cong T(S)$.

Thm. (for cubic 3-folds [Torelli]): X smooth cubic 3-fold $\Rightarrow$ $\Rightarrow T(X)$ is an abel. variety of dim. 5 ($\Rightarrow$ has an associated Θ-function. Θ-function on $X \Rightarrow \Theta$-divisor D of $J(X)$).

Thm. (Beauville): D has a unique singular pt S, which has mult. 3.

Thm.: $\mathbb{P}TC_{D,S} \cong X$, TC = tangent cone.

How to calculate the Hodge numbers?

$X \subseteq \mathbb{P}^N$ hypersurface, algebraic diff. forms on $\mathbb{P}$ with poles on X

Griffiths: Rey map $\rightsquigarrow$ alg. diff. forms on X.

$\mathbb{P}^N[z_0 : \ldots : z_N]$, $dV = dz_0 \wedge \ldots \wedge dz_N$, $E = \sum_{j=1}^{N} z_j \frac{\partial}{\partial z_j}$, $\Omega = i(E)\, dV$.

$X = V(f)$, $\Omega(A) = \frac{A\Omega}{f^q}$, A homogeneous pol. of deg. d $\Rightarrow$

$\Rightarrow \Omega(A)$ is homog. of deg. 0.

Thm.: Poincaré residues of such $\Omega(A)$ span $F^{N-q}H^{N-1}(X, \mathbb{C})$ and Rey $\Omega(A) \in F^{N-q+1}H^{N-1}(X,\mathbb{C}) \Leftrightarrow H \in$ Jacobian ideal of f.

Es.: X Fermat cubic hypersurface in $\mathbb{P}^4$,

$F^3H^3(X,\mathbb{C}) = F^{N-q}H^{N-1}(X,\mathbb{C}) \Rightarrow N=4$, $q=1 \Rightarrow \Omega(A) = \frac{A\Omega}{f}$,

of degree 0: $\deg f = 3$, $\deg \Omega = 5$ $\ddot{\frown}$.

$\dim H^{2,1} = \dim F^2/F^3 = 5$: $\Omega(A) = \frac{A\Omega}{f^2} \Rightarrow \deg A = 1 \Rightarrow$

$\Rightarrow \dim H^{2,1} = 5$.