Dato $\alpha \in (0,1)$ si dice che $f:[a,b] \to \mathbb{R}$ è α-Hölder continua se

$$[f]_{C^\alpha} = \sup_{\substack{s \neq t,\\ s,t \in [a,b]}} \frac{|f(t)-f(s)|}{|s-t|^\alpha} < +\infty.$$

Teo.: sia $(X_t)_{t\in[0,1]}$ processo stocastico reale t.c. $\exists\, C, \varepsilon, \gamma > 0$ t.c. $\mathbb{E}[|X_t - X_s|^\gamma] \leq C|t-s|^{1+\varepsilon}$ $\forall s,t \in [0,1]$.
Allora $\exists$ modificazione $(\tilde{X}_t)_{t\in[0,1]}$ t.c. $\mathbb{P}$-q.c. le traiettorie sono α-Hölder cont. su $[0,1]$ $\forall \alpha < \varepsilon/\gamma$ e $\mathbb{E}\left[[\tilde{X}]^\gamma_{C^\alpha}\right] < +\infty$.

Dim.: idea: $X_t - X_s = (X_t - X_{t_m}) + (X_{t_m} - X_{s_m}) + (X_{s_m} - X_t)$,
$s_m, t_m \in D_m = \{t = k2^{-m} \mid k = 0,1,\dots,2^m\}$, i diadici di stadio m-esimo.
$D = \bigcup_{m=1}^{+\infty} D_m$. Poniamo $\Delta_m \{(s,t) \in D_m^2 \mid |s-t| = 2^{-m}\}$.
$|\Delta_m| \leq 2^{m+1}\cdot 2$. Sia $K_m = \max_{(s,t)\in\Delta_m} |X_t - X_s| \Rightarrow \mathbb{E}[K_m^\gamma] \leq$

$$\leq \sum_{(s,t)\in\Delta_m} \mathbb{E}[|X_t - X_s|^\gamma] \leq C\cdot 2^{-m(1+\varepsilon)}\cdot 2^m \cdot 4 \leq C'\cdot 2^{-m\varepsilon}.$$

Siano $s,t \in D$ e t.c. $|s-t| \leq 2^{-n}$. Supponiamo che $s<t$.
$\forall m$ sia $s_m \in D_m$, $s_m \in \operatorname{argmin}\{|u-s| \mid u \in D_m, u \geq s\}$,
$t_m \in D_m$, $t_m \in \operatorname{argmin}\{|u-t| \mid u \in D_m, u \leq t\}$.
Se $m=n$, $s_n = t_n$ oppure $(s_n, t_n) \in \Delta_n$.
In generale $(s_{m+1}, s_m), (t_{m+1}, t_m) \in \Delta_{m+1}$ o sono uguali
Se m è sufficientemente grande, $s_m = s$ e $t_m = t$.

$$X_t - X_s = (X_t - X_{t_n}) + (X_{t_n} - X_{s_n}) + (X_{s_n} - X_s) =$$

$$= \left(\sum_{m\geq n} X_{t_{m+1}} - X_{t_m}\right) + (X_{t_n} - X_{s_n}) - \left(\sum_{m\geq n} X_{s_{m+1}} - X_{s_m}\right) \Rightarrow$$

$$\Rightarrow |X_t - X_s| \leq 2\sum_{m>n} K_m + K_n \quad \text{se } |t-s| \leq 2^{-n}.$$

$$\sup_{\substack{s,t\in D,\\ s\neq t}} \frac{|X_t - X_s|}{|s-t|^\alpha} \leq \sup_{n\geq 0} \sup_{\substack{s,t\in D,\\ 2^{-n-1}<|t-s|\leq 2^{-n}}} \frac{|X_t - X_s|}{(2^{-n-1})^\alpha} \leq$$

$$\leq \sup_{n\geq 0} \cancel{\sup_{\substack{s,t\in D,\\ 2^{-n-1}<|t-s|\leq 2^{-n}}}} \left(2\sum_{m>n} K_m + K_n\right) 2^{(n+1)\alpha} \leq$$

$$\leq \sup_{n\geq 0} 2\sum_{m\geq n} K_m\, 2^{(m+1)\alpha} \leq 2\sum_{m\geq 0} K_m\, 2^{(m+1)\alpha}.$$

Se $\gamma \geq 1$, $\mathbb{E}\left[\left(\sup_{\substack{s,t\in D,\\ s\neq t}} \frac{|X_t - X_s|}{|t-s|^\alpha}\right)^\gamma\right]^{1/\gamma} \leq 2\sum_{m\geq 0} \mathbb{E}[K_m^\gamma]^{1/\gamma}\, 2^{(m+1)\alpha} \leq$

$\leq 2C'\sum_{m\geq 0} 2^{-\frac{\varepsilon m}{\gamma}}\, 2^{(m+1)\alpha} \underbrace{\quad}_{<+\infty}$ se $\alpha < \varepsilon/\gamma$. [$\gamma<1$: non c'è l'$1/\gamma$ fuori]

$\exists\, N \subseteq \Omega$ trascurabile t.c. per $\omega \in N^c$

$$\sup_{\substack{s,t\in D,\\ s\neq t}} \frac{|X_t(\omega) - X_s(\omega)|}{|t-s|^\alpha} < +\infty \Rightarrow t \mapsto X_t(\omega) \text{ è unif.}$$

cont. $\Rightarrow$ $\tilde{X}_t(\omega) = \lim_{\substack{s\to t,\\ s\in D}} X_s(\omega)$ è ben def. $\forall t \in [0,1]$.

$(\tilde{X}_t)_{t\in[0,1]}$ è la modificazione cercata di $(X_t)_{t\in[0,1]}$.
$t\in[0,1]$, $X_t = \tilde{X}_t$ $\mathbb{P}$-q.c.? $\mathbb{P}$-q.c., se $s \in D$ $X_s = \tilde{X}_s \Rightarrow$
$\Rightarrow \lim_{\substack{s\in D,\\ s\to t}} X_s = \tilde{X}_t$ $\mathbb{P}$-q.c.. D'altra parte, per Fatou

$$\mathbb{E}[|X_t - \tilde{X}_t|^\gamma] = \mathbb{E}\left[\left|X_t - \lim_{\substack{s\to t,\\ s\in D}} X_s\right|^\gamma\right] =$$

$$= \mathbb{E}\left[\lim_{\substack{s\to t,\\ s\in D}} |X_t - X_s|^\gamma\right] \leq \liminf_{s\to t,\, s\in D} \underbrace{\mathbb{E}[|X_t - X_s|^\gamma]}_{\leq\, C|t-s|^{1+\varepsilon}} = 0. \quad \square$$

Cor.: il BM ammette una versione α-Hölder continua $\forall \alpha < 1/2$.

Dim.: $\mathbb{E}[|B_t - B_s|^\gamma] = \mathbb{E}\left[\left(\frac{|B_t - B_s|}{\sqrt{|t-s|}}\right)^\gamma\right]\cdot|t-s|^{\gamma/2} \leq \mathbb{E}[|Z|^\gamma]\cdot|t-s|^{\gamma/2}$,

$$\gamma/2 = 1+\varepsilon \Rightarrow \alpha < \varepsilon/\gamma = \frac{1}{2} - \frac{1}{\gamma}.\ \gamma \to +\infty \Rightarrow \alpha \to 1/2. \quad \square$$

Teo. (Levy): sia $h(t) = \sqrt{2t\log(1/t)}$, $t>0$. Allora

$$\mathbb{P}\left(\limsup_{u\to 0^+} \sup_{\substack{s,t\in[0,1],\\ 0<|t-s|<u}} \frac{|B_t - B_s|}{h(u)} = 1\right) = 1.$$

Oss.: useremo le seguenti disuguaglianze sulle gaussiane. Sia $Z \sim N(0,1)$,

$$\frac{z}{z^2+1}\frac{e^{-z^2/2}}{\sqrt{2\pi}} \leq \mathbb{P}(Z>z) \leq \frac{1}{z}\frac{e^{-z^2/2}}{\sqrt{2\pi}} \quad \forall z>0.$$

Dim. (del teo.): fissiamo $\delta, \varepsilon, \eta \in (0,1)$. Poniamo D_n come sopra e
$\Delta_n = \{(s,t) \in D_n^2 \mid 0 < |s-t| < 2^{-n(1-\delta)}\}$, $|\Delta_n| \leq 2^{n+1}\cdot 2^{n\delta}\cdot 2$.

$$\mathbb{P}\left(\max_{s,t\in\Delta_n} \frac{|B_t - B_s|}{h(|t-s|)} \geq 1+\varepsilon\right) \leq \sum_{(s,t)\in\Delta_n} \mathbb{P}\left(\frac{|B_t - B_s|}{\sqrt{2|t-s|\log\frac{1}{|t-s|}}} \geq 1+\varepsilon\right) =$$

$$= \sum_{(s,t)\in\Delta_n} \mathbb{P}\left(|\overset{\sim N(0,1)}{Z}| \geq (1+\varepsilon)\sqrt{2\log\frac{1}{|t-s|}}\right) \leq$$

$$\leq \sum_{(s,t)\in\Delta_n} \frac{1}{(1+\varepsilon)\sqrt{2\log\frac{1}{|t-s|}}}\cdot \exp\left(-\frac{1}{\cancel{2}}(1+\varepsilon)^2\cdot\cancel{2}\log\frac{1}{|t-s|}\right) \leq$$

$$\leq \sum_{(s,t)\in\Delta_n} \frac{C(\varepsilon,\delta)}{\sqrt{n}}\cdot 2^{(-(1+\varepsilon)^2(1-\delta)n)} \leq \frac{C(\varepsilon,\delta)}{\sqrt{n}}\, 2^{n[(1+\delta)-(1+\varepsilon)^2(1-\delta)]}.$$

$2^{-n(1-\delta)} \geq |t-s|$

Imponiamo $(1+\delta) < (1+\varepsilon)^2(1-\delta) \xRightarrow{\text{Borel-Cantelli}} \mathbb{P}$-q.c. $\exists\, n(\omega)$ t.c. $\forall n \geq n(\omega)$

$$\max_{(s,t)\in\Delta_n} \frac{|B_t - B_s|}{h(|t-s|)} < 1+\varepsilon.$$

Come in Kolmogorov, dati $s,t\in D$, $s<t$ consideriamo $s_n, t_n \in D_n$.
Se $t-s \leq 2^{-n(1-\delta)}$ scrivo $B_t - B_s = \sum_{m\geq n} B_{t_{m+1}} - B_{t_m} + B_{t_n} - B_{s_n} + \sum_{m\geq n} B_{s_m} - B_{s_{m+1}}$
con $(s_n, t_n) \in \Delta_n$. Se $n \geq n(\omega)$ otteniamo

$$|B_t - B_s| \leq \left[2\sum_{m\geq n} h(2^{-(m+1)}) + h(|t-s|)\right](1+\varepsilon).$$

vedi libro per i dettagli, serve un $\bar{n} = \bar{n}(\omega,\eta,\delta)$

$\forall \eta,\delta>0$ $\exists\, \bar{n} = \bar{n}(\delta,\eta)$ $\forall n \geq \bar{n}$ $\sum_{m\geq n} h(2^{-(m+1)}) \leq \eta\, h(2^{-(n+1)(1-\delta)})$.

Allora $|B_t - B_s| \leq \left[2\eta\, h(\underbrace{2^{-(n+1)(1-\delta)}}_{\leq\, |t-s|}) + h(|t-s|)\right](1+\varepsilon) \Rightarrow$

$$\Rightarrow \mathbb{P}\left(\limsup_{u\to 0^+} \sup_{\substack{s,t\in[0,1],\\ 0<|s-t|<u}} \frac{|B_t - B_s|}{h(u)} \leq 1+\varepsilon\right) = 1.$$

Per il viceversa sia $\delta>0$ e consideriamo

$$L_n = \mathbb{P}\left(\max_{1\leq k\leq 2^n} \frac{|B_{k2^{-n}} - B_{(k-1)2^{-n}}|}{h(2^{-n})} \leq 1-\delta\right) \overset{\text{incrementi indi.}}{=}$$

$$= \prod_{k=1}^{n} \mathbb{P}\left(\frac{|B_{k2^{-n}} - B_{(k-1)2^{-n}}|}{\sqrt{2^{-n}}} \leq (1-\delta)\sqrt{2\log 2^n}\right) =$$

$$= \mathbb{P}\left(|Z| \leq (1-\delta)\sqrt{2\log 2^n}\right)^{2^n} = \left(1 - \mathbb{P}\left(Z > (1-\delta)\sqrt{2\log 2^n}\right)\right)^{2^n} \leq$$

$$\leq \left(1 - \frac{2}{\sqrt{2\pi}}\left(\frac{(1-\delta)\sqrt{2\log 2^n}}{((1-\delta)\sqrt{2\log 2^n})^2 + 1}\right)\cdot 2^{-n(1-\delta)^2}\right)^{2^n} \leq$$

$$\underset{1-x\,\leq\, e^{-x}}{\leq} \exp\left(-\frac{2}{\sqrt{2\pi}}\frac{C}{\sqrt{n}}\, 2^{n(1-(1-\delta)^2)}\right) \underset{\text{velocemente}}{\xrightarrow{n\to+\infty}} 0 \xRightarrow{BC}$$

$\Rightarrow \mathbb{P}$-q.c. $\exists\, n(\omega)$ t.c. $\forall n \geq n(\omega)$

$$\max_{1\leq k\leq 2^n} \frac{|B_{k2^{-n}} - B_{(k-1)2^{-n}}|}{h(2^{-n})} > 1-\delta \Rightarrow$$

$\Rightarrow$ frequentemente $|B_{k2^{-n}} - B_{(k-1)2^{-n}}| > (1-\delta)h(2^{-n})$. $\square$