**Prop.:** siano $(M_n)_{n=0}^N$ submart. e $\lambda>0$. Allora

$$\mathbb{P}\left(\max_{n=0,\dots,N} M_n \geqslant \lambda\right) \leqslant \frac{\mathbb{E}[M_N \cdot \mathbb{1}_{\{\max_{n=0,\dots,N} M_n \geqslant \lambda\}}]}{\lambda}.$$

**Dim.:** poniamo $S=\mathrm{INF}\{n\in\{0,1,\dots,N\} \mid M_n\geqslant\lambda\}$. $S$ è t.d.a.:

$\{S>k\}=\bigcap_{n=0}^{k}\{M_n<\lambda\}\in\mathcal{F}_k$. $S$ è a valori in $\{0,1,\dots,N\}\cup\{+\infty\}$.

$\underset{\cap}{\{M_n<\lambda\}}$, $\mathcal{F}_n\subseteq\mathcal{F}_k$

arresto opzionale

$$T=N \Rightarrow \mathbb{E}[M_{S\wedge N}]\leqslant\mathbb{E}[M_N].$$

$$A=\{\max_{n=0,\dots,N} M_n\geqslant\lambda\}=\{S<+\infty\}.$$

$$\mathbb{E}[M_{S\wedge N}]=\mathbb{E}[M_{S\wedge N}\cdot\mathbb{1}_{\{S<+\infty\}}+M_{S\wedge N}\cdot\mathbb{1}_{\{S=+\infty\}}]=$$

$$=\mathbb{E}[M_S\cdot\mathbb{1}_{\{S<+\infty\}}+M_N\cdot\mathbb{1}_{\{S=+\infty\}}]\geqslant$$

$$\geqslant\lambda\mathbb{P}(S<+\infty)+\mathbb{E}[M_N\cdot\mathbb{1}_{\{S=+\infty\}}].$$

$$\mathbb{E}[M_N]=\mathbb{E}[M_N\cdot\mathbb{1}_{\{S<+\infty\}}+M_N\cdot\mathbb{1}_{\{S=+\infty\}}]\Rightarrow$$

$$\Rightarrow\lambda\mathbb{P}(S<+\infty)\leqslant\mathbb{E}[M_N\cdot\mathbb{1}_{\{S<+\infty\}}].\ \square$$

**Oss.:** prop. $\Rightarrow \lambda\mathbb{P}(\max_{n=0,\dots,N} M_n\geqslant\lambda)\leqslant\mathbb{E}[|M_N|\cdot\cancel{\mathbb{1}_{\{S<+\infty\}}}]$.

**Cor.:** sia $(M_n)_{n=0}^N$ una submart. positiva, o una mart.. Allora $\forall p\geqslant1\ \forall\lambda>0\ \mathbb{P}(\max_{n=0,\dots,N}|M_n|\geqslant\lambda)\leqslant\frac{\mathbb{E}[|M_N|^p]}{\lambda^p}$.

$$\forall p>1\ \mathbb{E}\left[\max_{n=0,\dots,N}|M_n|^p\right]^{1/p}\leqslant\frac{p}{p-1}\mathbb{E}[|M_N|^p]^{1/p}.$$

**Dim.:** se $M_n$ è mart., $|\cdot|$ convesso $\Rightarrow |M_n|$ submart. positiva.

Se $M_n$ è submart. positiva e $\mathbb{E}[M_N^p]<+\infty$, allora

$\forall n=0,\dots,N\ \mathbb{E}[M_n^p]<+\infty$: $0\leqslant M_n\leqslant\mathbb{E}[M_N\mid\mathcal{F}_n]\Rightarrow$

Jensen

$\Rightarrow M_n^p\leqslant\mathbb{E}[M_N^p\mid\mathcal{F}_n]$.

WLOG $M_n\in L^p$, allora $(|M_n|^p)_n$ è submart. $\Rightarrow$

$$\Rightarrow\forall c>0\ \mathbb{P}\left(\max_{n=0,\dots,N}|M_n|^p\geqslant c\right)\leqslant\frac{\mathbb{E}[|M_N|^p\cdot\mathbb{1}_{\{\max_{n=0,\dots,N}|M_n|^p\geqslant c\}}]}{c},\ c=\lambda^p.$$

$$\mathbb{P}(\underbrace{\max_{n=0,\dots,N}|M_n|}_{=M^*}\geqslant\lambda)\leqslant\frac{\mathbb{E}[|M_N|\cdot\mathbb{1}_{\{\max_{n=0,\dots,N}|M_n|\geqslant\lambda\}}]}{\lambda}.$$

$$\mathbb{E}[(M^*)^p]=\int_0^{+\infty}\mathbb{P}((M^*)^p>r)\,dr \overset{r=\lambda^p,\ dr=p\lambda^{p-1}d\lambda}{=}$$

$$=\int_0^{+\infty}\mathbb{P}(M^*>\lambda)\,p\lambda^{p-1}\,d\lambda\leqslant\int_0^{+\infty}\mathbb{E}[|M_N|\cdot\mathbb{1}_{\{M^*\geqslant\lambda\}}]\,p\lambda^{p-2}\,d\lambda=$$

$$\underset{\text{Fubini-Tonelli}}{=}\mathbb{E}\left[|M_N|\cdot p\cdot\int_0^{+\infty}\mathbb{1}_{\{M^*\geqslant\lambda\}}\lambda^{p-2}\,d\lambda\right]=$$

$$=\mathbb{E}\left[|M_N|\cdot p\cdot\int_0^{M^*}\lambda^{p-2}\,d\lambda\right]=\mathbb{E}\left[|M_N|\frac{p}{p-1}(M^*)^{p-1}\right]\leqslant$$

$$\underset{\text{Hölder } p,\frac{p}{p-1}}{\leqslant}\frac{p}{p-1}\mathbb{E}[|M_N|^p]^{1/p}\cdot\mathbb{E}[(M^*)^p]^{1-1/p}.$$

Per assicurare che il termine da dividere sia finito, si considera $M^*\wedge K$, $K>0$ cost., si manda $K\longrightarrow+\infty$ e si usa Beppo-Levi. $\square$

**Teo.** (disuguaglianze massimali di Doob): se $(X_t)_{t\in T}$ mart. cont. a dx, $T\subseteq\mathbb{R}$ intervallo, allora, posto $X^*=\sup_{t\in T}|X_t|$,

$$\forall p\geqslant1\ \forall\lambda>0\ \mathbb{P}(X^*\geqslant\lambda)\leqslant\sup_{t\in T}\frac{\mathbb{E}[|X_t|^p]}{\lambda^p},$$

$$\forall p>1\ \mathbb{E}[(X^*)^p]^{1/p}\leqslant\frac{p}{p-1}\sup_{t\in T}\mathbb{E}[|X_t|^p]^{1/p}.$$

**Dim.:** per $F\subseteq T$ finito poniamo $X_F^*=\sup_{t\in F}|X_t|$.

Se $D\subseteq T$ è un denso numerabile, scriviamo le disuguaglianze per $F_n=\{t_i\}_{i=1}^n$, $D=\{t_i\}_{i=1}^{+\infty}$.

Per Beppo-Levi otteniamo le disuguaglianze per $X_D^*$; per continuità a dx, $X_D^*=X^*$. $\square$

**Es.:** sia $(B_t)_{t\geqslant0}$ BM. Poniamo $S_t=\max_{0\leqslant s\leqslant t}B_s$. Allora $\forall a>0$ $\forall t\geqslant0$ $\mathbb{P}(S_t>at)\leqslant\exp(-a^2t/2)$.

Oss.: vedremo che $\mathbb{P}(S_t>\lambda)=\mathbb{P}(|B_t|>\lambda)\ \forall\lambda$.

Ricordiamo la mart. $M_t^\alpha=\exp(\alpha B_t-\alpha^2t/2)$, $\alpha>0$, $t\geqslant0$.

$\sup_{0\leqslant s\leqslant t}M_s^\alpha\geqslant\exp(\alpha S_t-\alpha^2t/2)$. Nell'evento $S_t>at$,

$$\sup_{0\leqslant s\leqslant t}M_s^\alpha\geqslant\exp(\alpha at-\alpha^2t/2)\Longrightarrow$$

$$\Rightarrow\mathbb{P}(S_t>at)\leqslant\mathbb{P}\left(\sup_{0\leqslant s\leqslant t}M_s^\alpha\geqslant\exp(\alpha at-\alpha^2t/2)\right)\leqslant$$

Doob

$$\leqslant\sup_{0\leqslant s\leqslant t}\underbrace{\mathbb{E}[M_t^\alpha]}_{=1}\cdot\exp(-\alpha at+\alpha^2t/2),\ \alpha=a.$$

**Teo.** (degli attraversamenti di Doob): dato un insieme di tempi $F=\{t_1<t_2<\dots<t_d\}$ e un processo $(M_t)_{t\in F}$, definiamo i seguenti t.d.a. ($a<b\in\mathbb{R}$): $S_1=\mathrm{INF}\{t\in F\mid M_t\leqslant a\}$, $S_2=\mathrm{INF}\{t\in F\mid t\geqslant S_1, M_t\geqslant b\}$ e analogamente $S_{2k+1}, S_{2k+2}$, alternando $a$ e $b$.

Oss.: $S_k=+\infty\ \forall k>d$.

Definiamo $U(M,F,[a,b])=\sum_{k=1}^{+\infty}\mathbb{1}_{\{S_{2k}<+\infty\}}$. → # attraversamenti in salita

Oss.: $D(M,F,[a,b])=U(-M,F,[-b,-a])$.

Se $T$ è infinito numerabile, $U((M_t)_{t\in T},T,[a,b])=$
$=\sup_{\substack{F\subseteq T\\ \text{finito}}}U((M_t)_{t\in F},F,[a,b])$.

Sia $(M_t)_{t\in T}$ una supermart., $T$ numerabile.

$$\mathbb{E}[U((M_t)_{t\in T},T,[a,b])]\leqslant\frac{\sup_{t\in T}\mathbb{E}[(M_t-a)^-]}{b-a}.$$

Oss.: $\sup_{t\in T}\mathbb{E}[(M_t-a)^-]\leqslant\sup_{t\in T}\mathbb{E}[|X_t|]+a$.

**Dim.:** WLOG basta il caso $T=F=\{t_1<\dots<t_d\}$ finito.

$\forall k=0,\dots,d$ applichiamo arresto opzionale ai tempi $S_{2k+1}\leqslant S_{2k+2}$, o meglio $S_{2k+1}\wedge t_d\leqslant S_{2k+2}\wedge t_d\leqslant t_d$:

$$\mathbb{E}[M_{S_{2k+1}\wedge t_d}]\geqslant\mathbb{E}[M_{S_{2k+2}\wedge t_d}]\Longrightarrow$$

$$\Rightarrow0\geqslant\mathbb{E}[M_{S_{2k+2}\wedge t_d}-M_{S_{2k+1}\wedge t_d}]=$$

$$=\mathbb{E}\Big[(M_{S_{2k+2}\wedge t_d}-M_{S_{2k+1}\wedge t_d})\cdot\mathbb{1}_{\{S_{2k+2}<+\infty\}}+$$

$$+(M_{S_{2k+2}\wedge t_d}-M_{S_{2k+1}\wedge t_d})\cdot\mathbb{1}_{\{S_{2k+2}=+\infty\}}\Big]\geqslant$$

$$\geqslant(b-a)\mathbb{E}[\mathbb{1}_{\{S_{2k+2}<+\infty\}}]+\mathbb{E}[(M_{t_d}-M_{S_{2k+1}})\cdot\mathbb{1}_{\{S_{2k+2}=+\infty,\ S_{2k+1}<+\infty\}}]\geqslant$$

$$\geqslant(b-a)\mathbb{E}[\mathbb{1}_{\{S_{2k+2}<+\infty\}}]-\mathbb{E}[(M_{t_d}-a)^-\cdot\mathbb{1}_{\{S_{2k+1}<S_{2k+2}=+\infty\}}].$$

Sommiamo per $k=0,\dots,d\Rightarrow$

$$\Rightarrow0\geqslant(b-a)\mathbb{E}[U(M,F,[a,b])]-\mathbb{E}\Big[(M_{t_d}-a)^-\cdot\sum_{k=0}^{d}\underbrace{\mathbb{1}_{\{S_{2k+1}<S_{2k+2}=+\infty\}}}_{A_k\text{ disgiunti}}\Big]\geqslant$$

$$\underset{\sum\mathbb{1}_{A_k}\leqslant1}{\geqslant}(b-a)\mathbb{E}[U(M,F,[a,b])]-\mathbb{E}[(M_{t_d}-a)^-].\ \square$$

**Teo.:** sia $(M_n)_{n\in\mathbb{N}}$ una submart. t.c. $\sup_n\mathbb{E}[|M_n|]<+\infty$. Allora $\mathbb{P}$-q.c. $\exists\lim_{n\to+\infty}M_n$ (finito $\mathbb{P}$-q.c.).

**Dim.:** poi. $\square$