Prop.: $(B_t)_{t\geq 0}$ BM, $Z=\{t\geq 0 \mid B_t=0\}$. Allora $\mathbb{P}$-q.c. Z è chiuso, ha parte interna vuota, senza pti isolati e $\mathcal{L}^1(Z)=0$.

Dim.: chiuso ok. $\mathbb{E}[\mathcal{L}^1(Z)]=\mathbb{E}\left[\int_0^{+\infty} \mathbb{1}_Z(t)\,dt\right]=$

$$=\int_0^{+\infty}\mathbb{E}[\mathbb{1}_Z(t)]\,dt=\int_0^{+\infty}\mathbb{P}(B_t=0)\,dt=0 \Rightarrow$$

$$\Rightarrow \mathcal{L}^1(Z)=0 \ \mathbb{P}\text{-q.c.} \Rightarrow \mathring{Z}=\emptyset \ \mathbb{P}\text{-q.c.}.$$

Dato $p\in\mathbb{Q}^+$, $d_p=\inf\{t\geq p \mid B_p=0\}$. Allora d_p è t.d.a. $\xRightarrow{\text{Markov forte}}$ $(B_{s+d_p}-\cancel{B_{d_p}})_{s\geq 0}$ è BM $\Rightarrow$

$$\Rightarrow \mathbb{P}\Bigg(\underbrace{\bigcap_{p\in\mathbb{Q}^+}\ \bigcap_{\substack{\varepsilon>0,\\ \varepsilon\in\mathbb{Q}}}\left\{\max_{s\leq\varepsilon} B_{s+d_p}>0,\ \min_{s\leq\varepsilon} B_{s+d_p}<0\right\}}_{\{Z \text{ non ha pti isolati}\}}\Bigg)=1.\ \square$$

Ex.: mostrare il principio di riflessione per $S_n=\sum_{i=1}^n X_i$, $n\in\mathbb{N}$, $X_i\in\{-1,1\}$ unif. e indi..

Dim. senza buchi del principio di riflessione:

sia $\tilde{B}_t=\begin{cases} B_t & \text{se } t<\tau_a \\ 2a-B_t & \text{se } t\geq\tau_a\end{cases}$. Supponiamo che $\tilde{B}$ sia BM.

Allora $\mathbb{P}(\tau_a\leq t)=\mathbb{P}(\tilde{B}_t>a,\tau_a\leq t)+\mathbb{P}(\tilde{B}_t\leq a,\tau_a\leq t)$.

$$\mathbb{P}(\tilde{B}_t>a,\tau_a\leq t)\underset{\{\tilde{B}_t>a\}\subseteq\{\tau_a\leq t\}}{=}\mathbb{P}(\tilde{B}_t>a)\underset{\tilde{B}\text{ BM}}{=}\mathbb{P}(B_t>a)\underset{B_t\sim\mathcal{N}(0,t)}{=}\mathbb{P}(B_t\geq a).$$

$$\mathbb{P}(\tilde{B}_t\leq a,\tau_a\leq t)=\mathbb{P}(2a-B_t\leq a,\tau_a\leq t)=$$
$$=\mathbb{P}(B_t\geq a,\tau_a\leq t)\underset{\{B_t\geq a\}\subseteq\{\tau_a\leq t\}}{=}\mathbb{P}(B_t\geq a).$$

Vediamo che $\tilde{B}$ è effettivamente un BM.

Lemma: siano $(X_t)_{t\geq 0}$ un processo continuo, $T:\Omega\to[0,+\infty)$, $\mathcal{F}\subseteq\mathcal{A}$, $(\Omega,\mathcal{A},\mathbb{P})$, U,T v.a. $\mathcal{F}$-mis., X indi. da $\mathcal{F}$. Allora

$$\mathbb{E}[\varphi(X_T,U)\mid\mathcal{F}]=\mathbb{E}[\varphi(X_t,u)]_{\substack{t=T,\\ u=U}},$$

$\forall\varphi: E\times F\to\mathbb{R}$ mis. e limitata.

Dim.: considero una successione $(T^k)_k$ di v.a. discrete t.c. $T^k\downarrow T$: $T_k=\begin{cases} a2^{-k} & \text{se } T\in[a2^{-k},(a+1)2^{-k}),\ a=0,1,\dots,2^{2k} \\ 2^k & \text{se } T\geq 2^k\end{cases}$

$$\varphi(X_{T_k},U)=\sum_t \varphi(X_t,U)\mathbb{1}_{\{T_k=t\}}\Rightarrow$$

$$\Rightarrow \mathbb{E}[\varphi(X_{T_k},U)\mid\mathcal{F}]=\sum_t \mathbb{E}[\varphi(X_t,u)]_{u=U}\mathbb{1}_{\{T_k=t\}}=$$

$$=\mathbb{E}[\varphi(X_t,u)]_{\substack{u=U,\\ t=T_k}}.\quad T_k\to T\Rightarrow\varphi(X_{T_k},U)\to\varphi(X_T,U)\xRightarrow{\text{con. dom.}}$$

$$\Rightarrow \mathbb{E}[\varphi(X_{T_k},U)\mid\mathcal{F}]\to\mathbb{E}[\varphi(X_T,U)\mid\mathcal{F}].$$ Inoltre, a u fissato $t\mapsto\mathbb{E}[\varphi(X_t,u)]$ è continua $\Rightarrow$

$$\Rightarrow \mathbb{E}[\varphi(X_t,u)]_{\substack{u=U,\\ t=T_k}}\to\mathbb{E}[\varphi(X_t,u)]_{\substack{u=U,\\ t=T}}.\ \square$$

$(B_t)_{t\geq 0}$ BM, $s<t$, $\varphi:\mathbb{R}\to\mathbb{R}$ cont. e lim., $A\in\mathcal{F}_s$.

$$\mathbb{E}[\varphi(\tilde{B}_t-\tilde{B}_s)\cdot\mathbb{1}_A]\overset{?}{=}\mathbb{E}[\varphi(B_t-B_s)]\cdot\mathbb{P}(A).$$

$$\mathbb{E}[\varphi(\tilde{B}_t-\tilde{B}_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{\tau_a>t\}}]=$$
$$=\mathbb{E}[\varphi(B_t-B_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{\tau_a>t\}}].$$

$$\mathbb{E}[\varphi(\tilde{B}_t-\tilde{B}_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{\tau_a\leq s\}}]=$$
$$=\mathbb{E}[\varphi(B_s-B_t)\cdot\underset{\mathcal{F}_s}{\mathbb{1}_A}\cdot\underset{\mathcal{F}_s}{\mathbb{1}_{\{\tau_a\leq s\}}}]=\mathbb{E}[\varphi(B_s-B_t)]\cdot\mathbb{P}(A,\tau_a\leq s)=$$
$$\underset{B_s-B_t\sim\mathcal{N}(0,t-s)}{=}\mathbb{E}[\varphi(B_t-B_s)]\cdot\mathbb{P}(A,\tau_a\leq s)=\mathbb{E}[\varphi(B_t-B_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{\tau_a\leq s\}}].$$

$$\mathbb{E}[\varphi(\tilde{B}_t-\tilde{B}_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{s<\tau_a\leq t\}}]=$$
$$=\mathbb{E}[\varphi(2a-B_t-B_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{s<\tau_a\leq t\}}]=$$
$$=\mathbb{E}[\varphi(2B_{\tau_a}-B_t-B_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{s<\tau_a\leq t\}}]=$$
$$=\mathbb{E}[\varphi((B_{\tau_a}-B_t)+(B_{\tau_a}-B_s))\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{s<\tau_a\leq t\}}]=\bigstar.$$

$B_{\tau_a}-B_t=-(B_{\tau_a+(t-\tau_a)}-B_{\tau_a})$. Ricordiamo $(\underbrace{B_{\tau_a+r}-B_{\tau_a}}_{-X_r})_{r\geq 0}$ è BM.

$$\bigstar=\mathbb{E}[\varphi(X_{t-\tau_a}+B_{\tau_a}-B_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{s<\tau_a\leq t\}}]=$$
$$=\mathbb{E}[\varphi(X_{t-\tau_a\vee s}+B_{\tau_a\vee s}-B_s)\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{s<\tau_a\leq t\}}]=$$
$$=\mathbb{E}[\mathbb{E}[\varphi(X_{t-\tau_a\vee s}+B_{\tau_a\vee s}-B_s)\mid\mathcal{F}_{\tau_a\vee s}]\cdot\mathbb{1}_A\cdot\mathbb{1}_{\{s<\tau_a\leq t\}}]\underset{\text{lemma}}{=}$$
$$=\mathbb{E}\Big[\mathbb{1}_A\mathbb{1}_{\{s<\tau_a\leq t\}}\mathbb{E}[\varphi(X_r+u)]_{\substack{u=B_{\tau_a}-B_s,\\ r=t-\tau_a}}\Big]=$$
$$=\mathbb{E}\Big[\mathbb{1}_A\mathbb{1}_{\{s<\tau_a\leq t\}}\mathbb{E}[\varphi(-X_r+u)]_{\substack{u=B_{\tau_a}-B_s,\\ r=t-\tau_a}}\Big],$$ e

si ripercorrono tutti i passaggi. $\square$