Dim. (di alcune cose mancanti della volta scorsa, e che $M_t^2-\langle M\rangle_t$ è unif. int.): $\forall\, T_n$ t.d.a.

$$\mathbb{E}[(M_t^{T_n})^2\cdot 1_{\{T_n>0\}}]=\mathbb{E}[M_0^2\cdot 1_{\{T_n>0\}}]+\mathbb{E}[\langle M\rangle_t^{T_n}\, 1_{\{T_n>0\}}].$$

Doob $\Rightarrow \mathbb{E}[\sup_{t\geq0}|M_t^{T_n}|^2\cdot 1_{\{T_n>0\}}]\leq 2\sup_t \mathbb{E}[|M_t^{T_n}|^2\cdot 1_{\{T_n>0\}}]$.

Se valgono i) e ii) della prop., per Fatou e BL

$$\mathbb{E}[M_t^2]\leq \mathbb{E}[M_0^2]+\mathbb{E}[\langle M\rangle_t].$$

$\sup_{t\geq0}|M_t^{T_n}|^2\leq\sup_{t\geq0}|M_t|^2$ e per $n\to+\infty$

$\sup_{t\geq0}|M_t^{T_n}|^2\uparrow\sup_{t\geq0}|M_t|^2\Rightarrow$

$\Rightarrow \mathbb{E}[\sup_{t\geq0}|M_t|^2]=\sup_n\mathbb{E}[\sup_{t\geq0}|M_t^{T_n}|^2\cdot 1_{\{T_n>0\}}]\leq$

$\leq 2\sup_n\sup_t\mathbb{E}[|M_t^{T_n}|^2\cdot 1_{\{T_n>0\}}]\leq 2(\mathbb{E}[M_0^2]+\mathbb{E}[\langle M\rangle_\infty])\Rightarrow$

$\Rightarrow M_t^{T_n}\cdot 1_{\{T_n>0\}}\to M_t$ per $n\to+\infty$,

$|M_t^{T_n}|\cdot 1_{\{T_n>0\}}\leq\sup|M_t|\in L^2\Rightarrow\forall\, s<t$

$\mathbb{E}[M_t^{T_n}\cdot 1_{\{T_n>0\}}\,|\,\mathcal{F}_s]=M_s^{T_n}\cdot 1_{\{T_n>0\}}$,

$\downarrow$ Lebesgue $\qquad\qquad \downarrow$

$\mathbb{E}[M_t\,|\,\mathcal{F}_s] \qquad\qquad M_s$

quindi M_t è mart. lim. in $L^2\Rightarrow M\in H^2$.

Viceversa, se $M\in H^2$ $\mathbb{E}[(M_t^{T_n})^2\cdot 1_{\{T_n>0\}}]=$

$=\mathbb{E}[(M_0)^2\cdot 1_{\{T_n>0\}}]+\mathbb{E}[\langle M\rangle_t^{T_n}\cdot 1_{\{T_n>0\}}]$.

$M\in H^2\overset{\text{Doob}}{\Rightarrow}\sup_{t\geq0}|M_t|^2\in L^1$, $(M_t^{T_n})^2\cdot 1_{\{T_n>0\}}\leq\sup_{t\geq0}|M_t|^2$.

Per $n\to+\infty$ $(M_t^{T_n})^2\cdot1_{\{T_n>0\}}\to M_t^2$. Fatou e BL $\Rightarrow$

$\Rightarrow \mathbb{E}[M_t^2]=\mathbb{E}[M_0^2]+\mathbb{E}[\langle M\rangle_t]$

$\|M\|_{H^2}^2=\mathbb{E}[M_0^2]+\mathbb{E}[\langle M\rangle_\infty]$.

$M\in H^2\Rightarrow |M_t^2-\langle M\rangle_t|\leq\sup_{s\geq0}|M_s|^2+\langle M\rangle_\infty\in L^1\Rightarrow$

$\Rightarrow$ è unif. int.. □

Siano $(K_s)_{s\geq0}$ un processo e $(M_s)_{s\geq0}$ mart. cont. e lim..

Costruiamo l'integrale stocastico tramite somme di Riemann-Stieltjes.

$\Delta\subseteq[0,t]$, $0=t_0<\dots<t_n=t$, $\sum_{i=1}^n K_{t_{i-1}}(M_{t_i}-M_{t_{i-1}})$. Se K è lim. e adattato otteniamo una mart. a tempi discreti.

Def.: <u>processi elementari</u>: $K_s=K_{-1}\cdot 1_0(s)+\sum_{i=0}^n K_i\cdot 1_{(t_i,t_{i+1}]}(s)$, $s\geq0$, K_i $\mathcal{F}_{t_i}$-mis. e lim. (K_{-1} $\mathcal{F}_0$-mis.). $K\in\mathcal{E}$.

Dato $K\in\mathcal{E}$, "definiamo" l'integrale stocastico (di Itô)

$$(K\cdot M)_t=\int_0^t K_s\, dM_s=\sum_{i=0}^{m-1}K_0(M_{t_{i+1}}-M_{t_i})+K_m(M_t-M_{t_m}) \text{ per } t_m\leq t<t_{m+1}.$$

$K\cdot M$ è mart. cont.. $M\in H^2\Rightarrow M\in H^2$:

$$\mathbb{E}[(K\cdot M)_t^2]=\sum_{i=0}^{m-1}\underbrace{\mathbb{E}[K_i^2(M_{t_{i+1}}-M_{t_i})^2]}_{\mathbb{E}[K_i^2\,\mathbb{E}[(M_{t_{i+1}}-M_{t_i})^2|\mathcal{F}_{t_i}]]}+\mathbb{E}[K_m^2\underbrace{(M_t-M_{t_m})^2}_{\text{simile}}].$$

$\mathbb{E}[(M_{t_{i+1}}-M_{t_i})^2|\mathcal{F}_{t_i}]=\mathbb{E}[M_{t_{i+1}}^2-M_{t_i}^2|\mathcal{F}_{t_i}]=$

$=\mathbb{E}[M_{t_{i+1}}^2-\langle M\rangle_{t_{i+1}}+(\langle M\rangle_{t_{i+1}}-\langle M\rangle_{t_i})|\mathcal{F}_{t_i}]-M_{t_i}^2+\langle M\rangle_{t_i}=$

$=\mathbb{E}[\langle M\rangle_{t_{i+1}}-\langle M\rangle_{t_i}|\mathcal{F}_{t_i}]\Rightarrow$

$\Rightarrow\mathbb{E}[(K\cdot M)_t^2]=\sum_{i=0}^{m-1}\mathbb{E}[K_i^2(\langle M\rangle_{t_{i+1}}-\langle M\rangle_{t_i})]+\mathbb{E}[K_m^2(\langle M\rangle_t-\langle M\rangle_{t_m})]=$

$=\mathbb{E}\left[\int_0^t K_s^2\, d\langle M\rangle_s\right]$ → isometria di Itô per processi elementari

$K\cdot M$ è mart. lim. (se $K\in\mathcal{E}$ e M mart. cont. lim.).

$\langle K\cdot M\rangle_t=\int_0^t K_s^2\, d\langle M\rangle_s$.

Data una mart. N cont. e lim.,

$\langle K\cdot M,N\rangle_t=\int_0^t K_s\, d\langle M,N\rangle_s$.

Ex.: $\langle M,N\rangle^T=\langle M^T,N^T\rangle=\langle M,N^T\rangle$.

Def.: sia $M\in H^2$. Poniamo $\mathcal{L}^2(M)$ l'insieme dei processi $(K_s)_{s\geq0}$ che sono progr. mis. e t.c. $\mathbb{E}\left[\int_0^{+\infty}K_s^2\, d\langle M\rangle_s\right]<+\infty$.

Es.: $(B_s^t)_{s\geq0}=(B_{s\wedge t})_{s\geq0}$, $\mathcal{L}^2(B^t)=$

$=\{(H_s)_{s\geq0}$ progr. mis. t.c. $\mathbb{E}\left[\int_0^t H_s^2\, ds\right]<+\infty\}$.

$L^2(M):=\mathcal{L}^2(M)/\underset{\text{equiv. q.o.}}{\sim}$: $K\sim K'\iff\mathbb{E}\left[\int_0^{+\infty}(K_s-K'_s)^2\, d\langle M\rangle_s\right]$.

$L^2(M)$ è spazio di H. con prodotto scalare

$(K,H)_{L^2(M)}=\mathbb{E}\left[\int_0^{+\infty}H_sK_s\, d\langle M\rangle_s\right]$.

Teo.: sia $M\in H^2$. Dato $K\in L^2(M)$ $\exists!$ $K\cdot M\in H_0^2$ t.c.

$\langle K\cdot M,N\rangle_t=\int_0^t K_s\, d\langle M,N\rangle_s$ $\mathbb{P}$-q.c. $\forall\, N\in H^2$.

Notazione: $(K\cdot M)_t=\int_0^t K_s\, dM_s$.

Dim.: unicità $K\cdot M,\widetilde{K\cdot M}\in H_0^2$.

$\langle K\cdot M-\widetilde{K\cdot M},N\rangle=0=\langle K\cdot M,N\rangle-\langle\widetilde{K\cdot M},N\rangle=$

$=K\cdot\langle M,N\rangle-K\cdot\langle M,N\rangle=0$, $N=K\cdot M-\widetilde{K\cdot M}\Rightarrow$

$\Rightarrow 0=\langle K\cdot M-\widetilde{K\cdot M}\rangle_t=0\Rightarrow\underbrace{\mathbb{E}[\langle K\cdot M-\widetilde{K\cdot M}\rangle_{+\infty}]}_{=\|K\cdot M-\widetilde{K\cdot M}\|_{H^2}}=0$.

Esistenza: sia $L:H_0^2\to\mathbb{R}$ (fissati $K\in\mathcal{L}^2(M)$ e $M\in H^2$) dato da $L(N)=\mathbb{E}\left[\int_0^{+\infty}K_s\, d\langle M,N\rangle_s\right]=\mathbb{E}[(K\cdot\langle M,N\rangle)_{+\infty}]$.

È ben def.: ↳ è lineare

$\mathbb{E}\left[\int_0^{+\infty}|K_s|\, dVar_1(\langle M,N\rangle)_s\right]<+\infty$?

Kunita-Watanabe, $H_s=1\Rightarrow$

$\Rightarrow\mathbb{E}\left[\int_0^{+\infty}|K_s|\, dVar_1(\langle M,N\rangle)_s\right]\leq\left(\mathbb{E}\left[\int_0^{+\infty}K_s^2\,\langle M\rangle_s\right]\right)^{1/2}\left(\mathbb{E}\left[\int_0^{+\infty}d\langle N\rangle_s\right]\right)^{1/2}\leq$

$\leq\|K\|_{L^2(M)}\cdot\|N\|_{H^2}\Rightarrow$

$\Rightarrow|L(N)|\leq\|K\|_{L^2(M)}\cdot\|N\|_{H^2}\Rightarrow L$ è cont..

Per Riesz $\exists\, K\cdot M\in H_0^2$ t.c. $L(N)=(N,K\cdot M)_{H^2}=$

$=\mathbb{E}[\underbrace{N_0\,(K\cdot M)_0}_{0}]+\mathbb{E}[\langle N,K\cdot M\rangle_\infty]=\lim_{t\to+\infty}\mathbb{E}[N_t(K\cdot M)_t]=\mathbb{E}[N_\infty(K\cdot M)_\infty]$

$=\mathbb{E}\left[\int_0^{+\infty}K_s\, d\langle M,N\rangle_s\right]=\mathbb{E}[(K\cdot\langle M,N\rangle)_\infty]$.

Verifichiamo che $\langle K\cdot M,N\rangle=K\cdot\langle M,N\rangle$, ossia $(K\cdot M)_tN_t-(K\cdot\langle M,N\rangle)_t$ è mart.: T t.d.a. lim., $\mathbb{E}[(K\cdot M)_TN_T-(K\cdot\langle M,N\rangle)_T]=0$:

$\mathbb{E}[(K\cdot M)_TN_T]=\mathbb{E}[\mathbb{E}[(K\cdot M)_TN_T\,|\,\mathcal{F}_T]]=$

$=\mathbb{E}[N_T\,\mathbb{E}[(K\cdot M)_T|\mathcal{F}_T]]=\mathbb{E}[N_T\,\mathbb{E}[(K\cdot M)_\infty|\mathcal{F}_T]]=$

$=\mathbb{E}[N_T(K\cdot M)_\infty]=\mathbb{E}[N_\infty^T(K\cdot M)_\infty]=(N^T,K\cdot M)_{H^2}=$

$=L(N^T)=\mathbb{E}\left[\int_0^{+\infty}K_s\, d\langle M,N^T\rangle_s\right]=\mathbb{E}\left[\int_0^{+\infty}K_s\, d\langle M,N\rangle_s^T\right]=$

$=\mathbb{E}\left[\int_0^T K_s\, d\langle M,N\rangle_s\right]=\mathbb{E}[(K\cdot\langle M,N\rangle)_T]$. □

Oss.: se $K\in\mathcal{E}$ è la def. di prima.