Teo. (P. Lévy): sia $(X_t)_{t\geq 0}$ proc. cont. e $(\mathcal{F}_t)$-adattato, $X_0=0$. TFAE:

i) X è un $(\mathcal{F}_t)_{t\geq 0}$-BM;

ii) X è mart. loc. cont. e $\langle X^i, X^j\rangle_t = \delta_{ij} t \ \forall t\geq 0 \ \forall i,j=1,\dots,d$;

iii) $\forall f_1,\dots,f_d \in L^2([0,+\infty),\mathcal{L}^1)$ i processi
$$\mathcal{E}_t^{if} := \exp\left(i\sum_{j=1}^d \int_0^t f_j(s)\,dX_s^j + \frac{1}{2}\sum_{j=1}^d \int_0^t f_j^2(s)\,ds\right)$$ sono mart. comp., e X è semimart. vettoriale cont.

Dim.: i) $\Rightarrow$ ii): già visto, per esercizio.

ii) $\Rightarrow$ iii): Itô $\Rightarrow \mathcal{E}_t^{if} = F\left(\left(\int_0^t f_j(s)\,dX_s^j, \int_0^t f_j^2(s)\,ds\right)_{j=1}^d\right)$,

$F: \mathbb{R}^{2d} \to \mathbb{C}$.

$(x_j,y_j)_{j=1}^d \mapsto \exp(i\sum_j x_j + \sum_j y_j/2)$

$$d\mathcal{E}_t^{if} = \sum_{j=1}^d \frac{\partial F}{\partial x^j} f_j\, dX_t^j + \frac{\partial F}{\partial y^j} f_j^2\, ds + \frac{1}{2}\sum_{j=1}^d \frac{\partial^2 F}{\partial (x^j)^2} f_j^2\, dt.$$

$$\frac{\partial F}{\partial x^j} = iF,\ \frac{\partial^2 F}{\partial (x^j)^2} = -iF,\ \frac{\partial F}{\partial y^j} = \frac{1}{2}F \Rightarrow d\mathcal{E}_t^{if} = i\sum_{j=1}^d F f_j\, dX_t^j.$$

$$|\mathcal{E}_t^{if}| \leq \left|\exp\left(i\sum_j\int_0^t f_j\,dX^j + \frac{1}{2}\sum_j\int_0^t f_j^2\,ds\right)\right| = \exp\left(\frac{1}{2}\int_0^t \sum_j f^2\,ds\right) \leq$$
$$\leq \exp\left(\frac{1}{2}\sum \|f_j\|_2^2\right) < +\infty.$$

iii) $\Rightarrow$ i): $\mathcal{E}_t^{if}$ è mart. (lim.), $s<t$. Tesi: $X_t - X_s \sim \mathcal{N}^d(0,t-s)$ indi. da $\mathcal{F}_s$. $A\in\mathcal{F}_s \Rightarrow \mathbb{E}[\mathcal{E}_t^{if}\mathbb{1}_A] = \mathbb{E}[\mathcal{E}_s^{if}\mathbb{1}_A]$. Sia $v\in\mathbb{R}^d$, $f_j(r) = v_j \mathbb{1}_{[s,t]}(r) \Rightarrow$ RHS $= \mathbb{P}(A)$, LHS $=$
$$= \mathbb{E}\left[\exp\left(i\sum_{j=1}^d v_j(X_t^j - X_s^j) + \frac{1}{2}\sum_{j=1}^d v_j^2(t-s)\right)\mathbb{1}_A\right] \Rightarrow$$
$$\Rightarrow \mathbb{P}(A) = \mathbb{E}\left[\mathbb{1}_A \exp(i\langle v, X_t - X_s\rangle)\right]\exp\left(\frac{1}{2}\|v\|^2(t-s)\right).$$
$$A=\Omega \Rightarrow \mathbb{E}[\exp(i\langle v, X_t-X_s\rangle)] = \varphi_{\mathcal{N}^d(0,t-s)}(v).$$

Lemma: se Y v.a. reale, $\mathcal{E}$ σ-algebra e $\forall A\in\mathcal{E}$
$$\mathbb{E}[\exp(iY\lambda)\mathbb{1}_A] = \mathbb{E}[e^{iY\lambda}]\mathbb{P}(A)\ \forall\lambda\in\mathbb{R} \Rightarrow$$
$\Rightarrow Y$ indi. da $\mathcal{E}$.

Dim.: ex.. $\square$

Lemma $\Rightarrow X_t - X_s$ indi. da $\mathcal{F}_s$. $\square$

Cor.: se $(X_t)_{t\geq 0}$ è mart. loc., $X_0=0$ e $\langle X\rangle_t = t$, allora X è BM.

Def.: il polinomio di Hermite di grado n è
$$h_n(x) = (-1)^n\exp\left(\frac{x^2}{2}\right)\frac{d^n}{dx^n}(\exp(-x^2/2)).$$
$$\sum_{n=0}^{+\infty}\frac{u^n}{n!}h_n(x) = \exp(ux - u^2/2)\ \forall u\in\mathbb{R}.$$
$$\frac{d}{dx}h_n(x) = n h_{n-1}(x).$$
$x\in\mathbb{R}, a>0$, definiamo $H_n(x,a) := h_n\left(\frac{x}{\sqrt{a}}\right)a^{n/2}$, $H_n(x,0) = x^n$.

Prop.: M mart. loc. cont., $M_0 = 0 \Rightarrow \forall n$ $H_n(M_t,\langle M\rangle_t)$ è mart. loc. cont. e vale $H_n(M_t,\langle M\rangle_t) = n!\int_0^t\left(\int_0^{s_{n-1}}\left(\int_0^{s_{n-2}}\cdots\left(\int_0^{s_1}dM_{s_0}\right)dM_{s_1}\dots dM_{s_{n-2}}\right)dM_{s_{n-1}}\right.$

Dim.: $f(M_t,\langle M\rangle_t)$ è mart. loc. se
$$\frac{1}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial y} \equiv 0.$$ H_n soddisfa. Segue anche
$$H_n(M_t,\langle M\rangle_t) = H_n(M_0,0) + \int_0^t \frac{\partial H_n}{\partial x}(M_s,\langle M\rangle_s)\,dM_s.$$
Induzione. $\square$

Ex.: X semimart. cont., $X_0 = 0$, $\mathcal{E}(X)_t = \exp(X_t - \langle X\rangle_t/2)$.
Allora $\mathcal{E}(X)$ è l'unica semimart. cont. Z t.c.
$$\begin{cases} Z_0 = 1 \\ dZ_t = Z_t\,dX_t \end{cases}.$$

Sol.: $\mathcal{E}(X)_t > 0\ \forall t$. A meno di arrestare al t.d.a. $T^\varepsilon = \inf\{t \mid \mathcal{E}(X_t)\leq\varepsilon\}$, possiamo applicare Itô a $f(Z_t,\mathcal{E}(X)_t)$, $f(x,y) = x/y$.

Oss.: $\frac{1}{\mathcal{E}(X)_t} = \exp(-X_t + \frac{1}{2}\langle X\rangle_t) = \mathcal{E}(-X)_t\exp(\langle X\rangle_t)$.

$\frac{Z_t}{\mathcal{E}(X)_t} = Z_t\,\mathcal{E}(-X)_t\exp(\langle X\rangle_t)$. Itô + conti $\Rightarrow$
$\Rightarrow d(Z_t\,\mathcal{E}(-X)_t\exp(\langle X\rangle_t)) = 0.$ $\square$