Teo. (disuguaglianze di Burkholder-Davis-Gundy): sia $(M_t)_{t\geqslant 0}$ mart. loc. cont. nulla in 0. Allora $\forall p>0\ \exists\, c_p\in(0,+\infty)$ t.c.
$$c_p^{-1}\mathbb{E}[\langle M\rangle_\infty^{p/2}]\leqslant \mathbb{E}[\sup_{t\geqslant 0}|M_t|^p]\leqslant c_p\mathbb{E}[\langle M\rangle_\infty^{p/2}].$$

Ex.: sia $(B_s)_{s\geqslant 0}$ BM e $(H_s)_{s\geqslant 0}\in\mathcal{L}^2_{loc}(B)$, $M_t=\int_0^t H_s\,dB_s$, $\langle M\rangle_t=\int_0^t H_s^2\,ds$. T t.d.a., allora
$$c_p^{-1}\mathbb{E}\left[\left(\int_0^T H_s^2\,ds\right)^{p/2}\right]\leqslant \mathbb{E}\left[\sup_{0\leqslant t\leqslant T}\left|\int_0^t H_s\,dB_s\right|^p\right]\leqslant$$
$$\leqslant c_p\mathbb{E}\left[\left(\int_0^T H_s^2\,ds\right)^{p/2}\right].$$

Oss.: se $p\geqslant 2$, $\left|\int_0^T H_s^2\,ds\right|^{p/2}\leqslant T^{p/2-1}\int_0^T H_s^p\,ds$. Se $T=t$ deterministico, $\mathbb{E}\left[\left|\int_0^t H_s^2\,ds\right|^{p/2}\right]\leqslant t^{p/2-1}\int_0^t \mathbb{E}[H_s^p]\,ds$.

Prop.: $p\geqslant 2\Rightarrow \exists\, c_p<+\infty$ t.c. $\mathbb{E}[\sup_{t\geqslant 0}|M_t|^p]\leqslant c_p\mathbb{E}[\langle M\rangle_\infty^{p/2}]$, M come sopra.

Dim.: prima il caso M mart. cont. e unif. lim..
$p\geqslant 2\Rightarrow f(x)=|x|^p\in C^2(\mathbb{R})$.
$f'(x)\ p|x|^{p-1}\mathrm{sgn}(x)$, $f''(x)=p(p-1)|x|^{p-2}$. Itô $\Rightarrow$
$$\Rightarrow d(f\circ M)_t=f'(M)\,dM_t+\frac{1}{2}f''(M)\,d\langle M\rangle_t\Rightarrow$$
$$\Rightarrow |M_t|^p=\int_0^t p|M_s|^{p-1}\mathrm{sgn}(M_s)\,dM_s+\frac{1}{2}\int_0^t p(p-1)|M_s|^{p-2}\,d\langle M\rangle_s.$$
$$\sup_{t\geqslant 0}|M_t|^p\leqslant p\sup_{t\geqslant 0}\left|\int_0^t|M_s|\,\mathrm{sgn}(M_s)\,dM_s\right|+$$
$$+\frac{p(p-1)}{2}\int_0^{+\infty}|M_s|^{p-2}\,d\langle M\rangle_s\Rightarrow$$
$$\Rightarrow \mathbb{E}[\sup_{t\geqslant 0}|M_t|^p]\leqslant p\,\mathbb{E}[\overbrace{\cdots}^{1}]+\frac{p(p-1)}{2}\mathbb{E}[\overbrace{\cdots}^{2}].$$

si può portare avanti, ma ci sono tanti conti

Oss.: $p\geqslant 2\underset{\text{Doob}}{\Rightarrow}\mathbb{E}[\sup_{t\geqslant 0}|M_t|^p]\leqslant\frac{p}{p-1}\sup_{t\geqslant 0}\mathbb{E}[|M_t|^p]$.
$$\mathbb{E}[|M_t|^p]=p\,\mathbb{E}\left[\int_0^t|M_s|^{p-1}\mathrm{sgn}(M_s)\,dM_s\right]+$$
$$+\frac{p(p-1)}{2}\mathbb{E}\left[\int_0^t|M_s|^{p-2}\,d\langle M\rangle_s\right].$$
$\mathbb{E}\left[\int_0^t|M_s|^{p-1}\mathrm{sgn}(M_s)\,dM_s\right]=0$ perché $t\mapsto\int_0^t|M_s|^{p-1}\mathrm{sgn}(M_s)\,dM_s\in H_0^2$ perché l'integranda è unif. lim. $\Rightarrow\in\mathcal{L}^2(M)$, e M è in H^2.
$$\mathbb{E}\left[\int_0^t|M_s|^{p-2}\,d\langle M\rangle_s\right]\leqslant\mathbb{E}\left[\int_0^t\left(\sup_{0\leqslant r\leqslant t}|M_r|^{p-2}\right)d\langle M\rangle_s\right]=$$
$$=\mathbb{E}\left[\left(\sup_{0\leqslant r\leqslant t}|M_r|^{p-2}\right)\langle M\rangle_t\right]\leqslant\mathbb{E}\left[\langle M\rangle_t^{p/2}\right]^{2/p}\mathbb{E}\left[(\cdots)^{\frac{p}{p-2}}\right]^{1-\frac{2}{p}}=$$
$$=\mathbb{E}[\langle M\rangle_t^{p/2}]^{2/p}\,\mathbb{E}\left[\sup_{0\leqslant r\leqslant t}|M_r|^p\right]^{1-2/p},$$
si semplifica e si manda $t\to+\infty$ applicando BL.
Caso generale: T_n t.d.a. $\uparrow+\infty$ t.c. $M^{T_n}\cdot 1\!\!1_{\{T_n>0\}}=M^{T_n}$ sono mart. cont. e lim. $\Rightarrow\mathbb{E}[\sup_{t\geqslant 0}|M_{T_n\wedge t}|^p]\leqslant c_p\mathbb{E}[\langle M^{T_n}\rangle_\infty]=$
$=c_p\,\mathbb{E}[\langle M\rangle_{T_n}]$ e si usa di nuovo BL. $\square$

Prop.: $p\geqslant 4\Rightarrow\exists\, c_p\in(0,+\infty)$ t.c. $\mathbb{E}[\langle M\rangle_\infty^{p/2}]\leqslant c_p\mathbb{E}[\sup_{t\geqslant 0}|M_t|^p]$.

Dim.: $f(x)=x^2\underset{\text{Itô}}{\Rightarrow}M_t^2-\langle M\rangle_t=2\int_0^t M_s\,dM_s$.
Caso M mart. unif. lim.:
$$\langle M\rangle_t=M_t^2-2\int_0^t M_s\,dM_s\Rightarrow$$
$$\Rightarrow\langle M\rangle_t^{p/2}=\left|M_t^2-2\int_0^t M_s\,dM_s\right|^{p/2}\leqslant$$
$$\leqslant c(p)|M_t|^p+c(p)\,2^{p/2}\left|\int_0^t M_s\,dM_s\right|^{p/2}$$
$$\mathbb{E}\left[\left|\int_0^t M_s\,dM_s\right|^{p/2}\right]\leqslant\mathbb{E}\left[\sup_{0\leqslant r\leqslant t}\Big|\overbrace{\int_0^r M_s\,dM_s}^{N_r}\Big|^{p/2}\right]\underset{\substack{p/2\geqslant 2 \text{ e}\\ \text{prop. precedente}}}{\leqslant}$$
$$\leqslant\sup_{0\leqslant s\leqslant t}C(p)\,\mathbb{E}[\langle N\rangle_s^{p/4}].$$
Oss.: $\langle N\rangle_t=\int_0^t M_s^2\,d\langle M\rangle_s$.
$$\mathbb{E}\left[\left(\int_0^t M_s^2\,d\langle M\rangle_s\right)^{p/4}\right]\leqslant\mathbb{E}\left[\left(\sup_{0\leqslant s\leqslant t}|M_s|^{p/2}\right)\langle M\rangle_t^{p/4}\right]\overset{CS}{\leqslant}$$
$$\leqslant\mathbb{E}[\underbrace{\langle M\rangle_t^{p/2}}_{1}]^{1/2}\,\mathbb{E}[\underbrace{\sup_{0\leqslant s\leqslant t}|M_s|^p}_{2}]^{1/2}.$$ Allora
$$\mathbb{E}[\langle M\rangle_t^{p/2}]\leqslant\tilde{c}(p)\,\mathbb{E}[\sup_{0\leqslant s\leqslant t}|M_s|^p]+\tilde{C}(p)\,\mathbb{E}[1]^{1/2}\,\mathbb{E}[2]^{1/2}\leqslant$$
$$\leqslant\tilde{c}(p)\,\mathbb{E}[\sup_{0\leqslant s\leqslant t}|M_s|^p]+\underbrace{\tilde{C}(p)\frac{\varepsilon}{2}}_{1/2\ \text{(scelgo }\varepsilon)}\mathbb{E}[1]+\frac{\tilde{C}(p)}{2\varepsilon}\mathbb{E}[2].$$
Si conclude come prima. $\square$