Sia χ_q (modulo q), $\chi_q \neq \chi_0$ e primitivo, cioè di periodo q.

$$\xi(s, \chi_q) = (\pi/q)^{-\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi_q)$$

$$a = \begin{cases} 0 & \text{se } \chi(-1) = 1 \\ 1 & \text{se } \chi(1) = -1 \end{cases}$$

$$L(s, \chi_q) = \sum_{n=1}^{+\infty} \frac{\chi(n)}{n^s}, \ \sigma > 0. \qquad \left|\sum_{n \leq x} \chi(n)\right| \leq q$$

$$\sum_{n=1}^{+\infty} \frac{|\chi(n)|}{n^s} \leq \sum_{n=1}^{+\infty} \frac{1}{n^s} = \zeta(s) \quad (\sigma > 1)$$

$$\xi(s, \chi_q) = \square \ \xi(1-s, \overline{\chi}_q)$$
$$|\square| = 1$$

C'è una corrispondenza $\rho_\chi \longleftrightarrow \overline{\rho}_{\overline{\chi}}$.

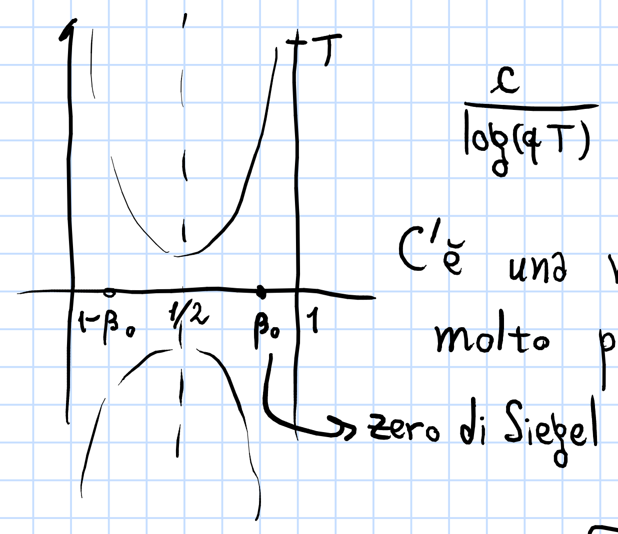
modulo q, $\chi_0(n) = \begin{cases} 1 & \text{se } (n, q) = 1 \\ 0 & \text{se } (n, q) > 1 \end{cases}$

$$\sigma > 1, \ L(s, \chi_0) = \prod_p \left(1 - \frac{\chi_0(p)}{p^s}\right)^{-1} = \prod_{p \nmid q} \left(1 - \frac{1}{p^s}\right)^{-1} =$$
$$= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \prod_{p|q} \left(1 - \frac{1}{p^s}\right) = \zeta(s) \prod_{p|q} \left(1 - \frac{1}{p^s}\right).$$

Residuo in $s=1$: $\prod_{p|q} \left(1 - \frac{1}{p}\right) = \frac{\phi(q)}{q}$.

$$\psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n) \Rightarrow \psi(x, q, a) = \frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\, q)} \overline{\chi}(a) \sum_{n \leq x} \Lambda(n) \chi(n) =$$
$$= \frac{1}{\phi(q)} \sum_{\chi (\mathrm{mod}\, q)} \overline{\chi}(a) \psi(x, \chi) = \frac{\psi(x, \chi_0)}{\phi(q)} + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi}(a) \psi(x, \chi).$$

$\chi \neq \chi_0$, χ primitivo $\Rightarrow \psi(x, \chi_q) = -\sum_{\rho_\chi} \frac{x^{\rho_\chi}}{\rho_\chi} + (\dots)$



$$\frac{c}{\log(qT)}$$

C'è una versione della RH molto più incasinati.

$$\psi(x, q, a) = \frac{x}{\phi(q)} + O\left(x \exp(-c\sqrt{\log x})\right)$$

A seconda dei casi, $q \leq (\log x)^A$, $A > 0$ o $q \leq (\log x)^{1-\delta}$.

Possibili zeri per $t=0$: zeri di Siegel. Ce n'è al più uno in una zona precisa $\left(\frac{c'}{\log q}\right)$.

Per $q \leq \frac{\sqrt{x}}{(\log x)^B}$, "in media" vale RH.

$\theta = \sup \beta \longrightarrow \psi(x) = x + O(x^\theta \log^2 x)$

RH $\longrightarrow \theta = 1/2$.

GRH (Grand RH): $\psi(x, q, a) = \frac{x}{\phi(q)} + O(x^{1/2} \log^2 x)$.

$\left|\psi(x, q, a) - \frac{x}{\phi(q)}\right| = O\left(x^{1/2} (\log x)^A\right)$ per quasi ogni $q \leq \frac{\sqrt{x}}{(\log x)^B}$.

Lemma (Mittag-Leffler): sia $z_1, z_2, \dots, z_n, \dots \longrightarrow +\infty$ t.c. $0 < |z_1| \leq |z_2| \leq \dots$. Sia $(m_n)_{n \in \mathbb{N}}$, $m_n \in \mathbb{C}^*$. $\exists\, p_n \in \mathbb{N} \cup \{0\}$ t.c. $f(z) = \sum_{n=1}^{+\infty} \left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}$ converge totalmente in $K \subset \mathbb{C} \setminus \{z_1, z_2, \dots\}$ cpt. Inoltre, se $|z| < |z_1|$, si ha

$$f(z) = -\sum_{k=1}^{+\infty} \left(\sum_{n,\ p_n < k} m_n z_n^{-k}\right) z^{k-1}.$$

Dim.: sia $0 < r_1 \leq r_2 \dots \longrightarrow +\infty$ t.c. $r_n < |z_n|$. Per $|z| \leq r_n$,

$$\left|\frac{m_n}{z - z_n}\right| \leq \frac{|m_n|}{|z_n| - r_n}, \quad \left|\frac{z}{z_n}\right| < \frac{|z|}{r_n} \leq 1 \Rightarrow$$

$$\exists\, p_n \in \mathbb{N} \cup \{0\} \text{ t.c. } \left|\frac{z}{z_n}\right|^{p_n} < \varepsilon_n \frac{|z_n| - r_n}{|m_n|}, \ \varepsilon_n > 0, \ \sum_{n=1}^{+\infty} \varepsilon_n < +\infty \Rightarrow$$

$$\Rightarrow \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| \leq \left|\frac{z}{z_n}\right|^{p_n} \frac{|m_n|}{|z_n| - r_n} < \varepsilon_n.$$

K, $z_1, z_2, \dots, z_N$, voglio $|z| \leq r_N \ \forall\, z \in K$.

Pongo $M_n = \max_{z \in K} \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right|$, $n \leq N-1$.

$$\sum_{n=1}^{+\infty} \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| \leq \sum_{n=1}^{N-1} M_n + \varepsilon_N + \varepsilon_{N+1} + \dots < +\infty \ \forall z \in K.$$

Se $|z| < |z_1|$, $f(z) = \sum_{n=1}^{+\infty} \left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n} = -\sum_{n=1}^{+\infty} \frac{m_n z^{p_n}}{z_n^{p_n+1}} \frac{1}{1 - \frac{z}{z_n}} =$

$$= -\sum_{n=1}^{+\infty} m_n \sum_{k=p_n+1}^{+\infty} \frac{z^{k-1}}{z_n^k} = -\sum_{k=1}^{+\infty} z^{k-1} \left(\sum_{n,\ p_n < k} m_n z_n^{-k}\right). \ \square$$

Oss.: $\left|\frac{z}{z_n}\right| = \frac{2|z|}{|z_n| + |z_n|} < \frac{2|z|}{|z_n| + r_n} \leq \frac{2|z|}{|z - z_n|} \Rightarrow$

$$\Rightarrow |m_n| \left|\frac{z}{z_n}\right|^{p_n+1} \leq \frac{2|z|}{|z - z_n|} \left|\frac{z}{z_n}\right|^{p_n} |m_n| = 2|z| \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{+\infty} |m_n| \left|\frac{z}{z_n}\right|^{p_n+1} \leq 2|z| \sum_{n=1}^{+\infty} \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| < +\infty.$$

Es.: se $z_n = n$ e $m_n = 1 \ \forall n$, basta $p_n = 1$.

Sia $|z_0| > \max_K |z|$, consideriamo $|z_n| > |z_0| + 1 \Rightarrow$

$$\Rightarrow \left|\left(\frac{z}{z_n}\right)^{p_n} \frac{m_n}{z - z_n}\right| \leq \left|\frac{z_0}{z_n}\right|^{p_n+1} \frac{|m_n|}{|z_n| - |z_0|} \frac{|z_n|}{|z_0|} \leq$$

$$\leq \left|\frac{z_0}{z_n}\right|^{p_n+1} \frac{|z_0| + 1}{\cancel{|z_0|} + 1 - \cancel{|z_0|}} \frac{|m_n|}{|z_0|} =$$

$\frac{t}{t - |z_0|}$ è decrescente

$= |m_n| \left|\frac{z_0}{z_n}\right|^{p_n+1} \left(1 + \frac{1}{|z_0|}\right)$, dunque per z_n siffatti vale anche la maggiorazione opposta.

Lemma: f meromorfa con poli semplici nei punti $z_n \neq 0$, $|z_n| \longrightarrow +\infty$, con residui $m_n \in \mathbb{Z} \ \forall n \in \mathbb{N}$. Allora la funzione

$$\varphi(z) = \exp \int_{\gamma(0,z)} f(w)\, dw \text{ è meromorfa, con}$$

zeri z_n con molteplicità m_n se $m_n > 0$ e poli z_n con molteplicità $-m_n$ se $m_n < 0$.

Oss.: γ' t.c. $\gamma' \cup -\gamma$ è semplice (piana) (di Jordan) $\subset \mathbb{C} \setminus \{z_1, \dots\}$.

$$\int_\gamma f(w)\, dw = \int_{\gamma'} f(w)\, dw + 2\pi i R, \quad R = \sum_{n,\ z_n \in A,\ \partial A = \gamma \cup \gamma'} m_n.$$

$$\varphi(z) = \exp\left(\int_{\gamma'} f(w)\, dw\right) e^{2\pi i R}.$$