Dim. (di Hadamard): $\nu = \lfloor \alpha \rfloor$. Vogliamo $D^{\nu+1} G \equiv 0$.

$$\log F(z) = G(z) + \sum_m \log\left(1-\frac{z}{z_m}\right) + \sum_{k=1}^{p} \frac{1}{k}\left(\frac{z}{z_m}\right)^k$$

derivo: $$\frac{F'}{F}(z) = G'(z) - \sum_m \frac{1}{z_m - z} + D\left(\sum_{k=1}^p \frac{1}{k} \left(\frac{z}{z_m}\right)^k\right)$$

$$D^\nu\left(\frac{F'}{F}(z)\right) = G^{(\nu+1)}(z) - \nu! \sum_m \frac{1}{(z_m-z)^{\nu+1}} + 0 \rightarrow p \le \alpha$$

Fissiamo $R>0$, sia $\varphi_R(z) = \frac{F(z)}{F(0)} \prod_{|z_m|\le R}\left(1-\frac{z}{z_m}\right)^{-1}$.

$|z|=2R$, $\left|1-\frac{z}{z_m}\right| \ge \left|\frac{z}{z_m}\right| - 1 \ge 2-1 = 1 \Rightarrow$

$\Rightarrow |\varphi_R(z)| \le \frac{|F(z)|}{|F(0)|} \ll_\varepsilon e^{(2R)^{\alpha+\varepsilon}} \quad \forall \varepsilon>0$

Principio del max. $\Rightarrow \forall\ |z|\le 2R$.

$\log|\varphi_R(z)| \le C(\varepsilon) R^{\alpha+\varepsilon}$

$\psi_R(z) = \log(\varphi_R(z))$ in $|z|\le R$ è olo. e $\psi_R(0)=0$.

BC $\Rightarrow \max_{|z|=\frac{R}{2}} |\psi_R^{(\nu+1)}(z)| \ll_{\varepsilon,\nu} \frac{R}{R^{\nu+2}} R^{\alpha+\varepsilon} \ll R^{\alpha-\nu-1+\varepsilon}$.

Sia $\varepsilon>0$ t.c. $\alpha-\nu-1+\varepsilon<0$.

$$\psi_R'(z) = \frac{\varphi_R'(z)}{\varphi_R(z)} = \frac{F'(z)}{F(z)} + \sum_{|z_m|\le R}\frac{1}{z_m - z}$$

$$\psi_R^{(\nu+1)}(z) = D^\nu\left(\frac{F'}{F}(z)\right) + \sum_{|z_m|\le R} \frac{\nu!}{(z_m-z)^{\nu+1}} = G^{(\nu+1)}(z) - \sum_{|z_m|>R}\frac{\nu!}{(z_m-z)^{\nu+1}}$$

WLOG $|z|=R/2 \Rightarrow \left|\sum_{|z_m|>R}\frac{\nu!}{(z_m-z)^{\nu+1}}\right| \le C_1(\nu)\sum_{|z_m|>R}\frac{1}{|z_m|^{\nu+1}} = o(1)$ per $R\to+\infty \Rightarrow$

$\Rightarrow |G^{(\nu+1)}(z)| \le C(\nu,\varepsilon) R^{-\delta} + o(1)$. Si fissa z e si manda $R\to+\infty$. □

Dim. (dell'altro Teo.): sia $F(z) = \prod_m E\left(\frac{z}{z_m}, p\right)$. Voglio

$$\log\left|\prod_m E\left(\frac{z}{z_m},p\right)\right| = \sum_m \log\left|E\left(\frac{z}{z_m},p\right)\right| \le C(\varepsilon)|z|^{\beta+\varepsilon} \quad \forall\varepsilon>0.$$

1) $\left|\frac{z}{z_m}\right|\le 1/2 \Rightarrow \log\left|E\left(\frac{z}{z_m},p\right)\right| \le \left|\log E\left(\frac{z}{z_m},p\right)\right| =$

$= \left|\log\left(1-\frac{z}{z_m}\right) + \sum_{k=1}^p \frac{1}{k}\left(\frac{z}{z_m}\right)^k\right| = \left|\sum_{k=p+1}^{+\infty}\frac{1}{k}\left(\frac{z}{z_m}\right)^k\right| \le$

$\le \sum_{k=p+1}^{+\infty}\left|\frac{z}{z_m}\right|^k \le \left|\frac{z}{z_m}\right|^{p+1}\left(1+\frac{1}{2}+\frac{1}{4}+\dots\right) = 2\left|\frac{z}{z_m}\right|^{p+1}$

$$\sum_{|z_m|\ge 2|z|}\log\left|E\left(\frac{z}{z_m},p\right)\right| \le 2|z|^{p+1}\sum_{|z_m|\ge2|z|}\frac{1}{|z_m|^{p+1}} = \begin{cases} 2|z|^\beta \sum_{|z_m|\ge 2|z|} \frac{1}{|z_m|^{p+1}} \le C|z|^\beta & \text{se } \beta=p+1 \\ (\dots) & \end{cases}$$

$(\dots)$: se $\beta<p+1$, prendo $\varepsilon<p+1-\beta$ e viene

$$2|z|^{\beta+\varepsilon}\sum_{|z_m|\ge2|z|}\frac{1}{|z_m|^{p+1}}|z|^{p+1-\beta-\varepsilon} \le \frac{\cancel{2}|z|^{\beta+\varepsilon}}{2^{p+\cancel{1}-\beta-\varepsilon}}\sum_{|z_m|\ge2|z|}\frac{1}{|z_m|^{\cancel{p+1}+\beta+\varepsilon-\cancel{p-1}}} \le$$

$\le C(\varepsilon)|z|^{\beta+\varepsilon}$

2) $\left|\frac{z}{z_m}\right|>\frac{1}{2}$, $\log\left|E\left(\frac{z}{z_m},p\right)\right| = \log\left|1-\frac{z}{z_m}\right| + \Re\left(\sum_{k=1}^p \frac{1}{k}\left(\frac{z}{z_m}\right)^k\right) \le$

$$\le \log\left(1+\left|\frac{z}{z_m}\right|\right) + \sum_{k=1}^p \left|\frac{z}{z_m}\right|^k \le \begin{cases} C_1(\varepsilon)\left|\frac{z}{z_m}\right|^\varepsilon \ \forall\varepsilon>0 & \text{se } p=0\\ C\left|\frac{z}{z_m}\right|^p & \text{se } p\ge1\end{cases}$$

$p=0$: $\sum_{|z_m|<2|z|}\left|\log\left|E\left(\frac{z}{z_m},p\right)\right|\right| \le C_1(\varepsilon)|z|^\varepsilon\sum_{|z_m|<2|z|}|z_m|^{-\varepsilon} = C_1(\varepsilon)|z|^{\beta+\varepsilon}\sum_{|z_m|>2|z|}|z_m|^{-\varepsilon}|z|^{-\beta} \le$

$\le C_1(\varepsilon)2^\beta|z|^{\beta+\varepsilon}\sum_{|z_m|<2|z|}\frac{1}{|z_m|^{\beta+\varepsilon}} \le C_2(\varepsilon)|z|^{\beta+\varepsilon}$.

$p\ge1$: $\sum_{|z_m|<2|z|}\log\left|E\left(\frac{z}{z_m},p\right)\right| \le C|z|^p\sum_{|z_m|<2|z|}|z_m|^{-p} =$

$= C_4|z|^{\beta+\varepsilon}\sum_{|z_m|<2|z|}|z_m|^{-p}|z|^{p-\beta-\varepsilon} \le C_4 2^{\beta+\varepsilon-p}|z|^{\beta+\varepsilon}\sum_{|z_m|<2|z|}\frac{1}{|z_m|^{\beta+\varepsilon}} \le$

$\le C_5(\varepsilon)|z|^{\beta+\varepsilon}$. □

Def.: data F intera, il genere è $g=\max\{p,q\}$.

Oss.: $\alpha-1\le g\le\alpha$. $g\le\alpha$ ovvio. Se $g<\alpha-1$, $q<\alpha-1$ e $p<\alpha-1$, ma $\alpha=\max\{\beta,q\}$. $p<\alpha-1\Rightarrow p+1<\alpha\Rightarrow\beta\le p+1<\alpha$, assurdo.

Es.: $F(z)=\frac{SIN(\pi z)}{\pi z}$. $SIN(\pi z) = \frac{e^{i\pi z}-e^{-i\pi z}}{2i} \ll_\varepsilon e^{|z|^{1+\varepsilon}}\ \forall\varepsilon>0\Rightarrow$

$\Rightarrow \alpha\le1$. $z_n=\pm n\ \forall n\in\mathbb{N}\Rightarrow\sum_{m=1}^{+\infty}\frac{1}{|z_m|^{1+\varepsilon}}<+\infty\ \forall\varepsilon>0\Rightarrow\beta=1,\ \alpha\ge\beta\Rightarrow$

$\Rightarrow\alpha=1$. $p=1$.

$$E\left(\frac{z}{n},1\right) = \left(1-\frac{z}{n}\right)\exp\left(\frac{z}{n}\right)$$

$E\left(\frac{z}{n},1\right)\cdot E\left(\frac{z}{-n},1\right) = \left(1-\frac{z^2}{n^2}\right)$. Ho $\prod_n\left(1-\frac{z^2}{n^2}\right)$. G?

$G(z) = \log F(0) + \frac{F'}{F}(0)z =$ ($\deg G\le1$. Altri termini di deg$=1$ no, lo vediamo dopo)

$= \log 1 + 0 = 0$

Allora $SIN(\pi z) = \pi z\prod_{m=1}^{+\infty}\left(1-\frac{z^2}{m^2}\right)$.

$\log(F(z)) = \log\left(\frac{SIN(\pi z)}{\pi z}\right) = \log(SIN(\pi z)) - \log(\pi z)$

derivo: $\frac{F'}{F}(z) = \pi\frac{\cos(\pi z)}{SIN(\pi z)} - \frac{1}{z} = \pi\,COTAN(\pi z) - \frac{1}{z}$.

Cor.: si ha $G(z) = \log F(0) + \sum_{k=1}^{q} D^{k-1}\left(\frac{F'}{F}(z)\right)_{z=0}\frac{z^k}{k!}$ se $q\le p$;

$G(z) = \log F(0) + \sum_{k=1}^p D^{k-1}\left(\frac{F'}{F}(z)\right)_{z=0}\frac{z^k}{k!} + \sum_{k=p+1}^{q}\left(\sum_m z_m^{-k}\right)\frac{z^k}{k}$ se $p<q$.

Inoltre, se $k>\max\{p,q\}\Rightarrow\sum_m z_m^{-k} = -\frac{1}{(k-1)!}D^{k-1}\left(\frac{F'}{F}(z)\right)_{z=0}$.