Dim. (del Cor. di prima): per un vecchio Cor.,

$$G(z)=\log F(0)+\sum_{k=1}^{+\infty}\left(\frac{1}{k!}D^{k-1}\left(\frac{F'}{F}(z)\right)_{z=0}+\frac{1}{k}\sum_{p_n<k} m_n z_n^{-k}\right)z^k$$

Se $q\leq p$, poiché G é pol. di $\deg=q$, $k\leq q\leq p \Rightarrow$

$\Rightarrow$ non c'é $\sum_{p_n<k}(\dots)$.

Se $q>p$, abbiamo il termine $\frac{1}{k}\sum_n z_n^{-k}$ per $p<k\leq q$,

cioé $\sum_{k=p+1}^{q}\frac{1}{k}\left(\sum_n z_n^{-k}\right)z^k$.

Se $k>\max\{p,q\}$, $0=\frac{1}{k!}D^{k-1}\left(\frac{F'}{F}(z)\right)_{z=0}+\frac{1}{k}\sum_n z_n^{-k}\Rightarrow$ tesi. □

Torniamo a $F(z)=\frac{\sin(\pi z)}{\pi z}$. $k\geq 2$, dal Cor.

$$\sum_n\frac{1}{n^k}+\sum_n\frac{1}{(-n)^k}=-\frac{D^{k-1}(\pi\cot(\pi z)-1/z)_{z=0}}{(k-1)!}$$

$k\geq 1$,

$$\zeta(2k)=\sum_{n=1}^{+\infty}\frac{1}{n^{2k}}=-\frac{D^{2k-1}\left(\frac{\pi}{2}\cot(\pi z)-\frac{1}{2z}\right)_{z=0}}{(2k-1)!}\Rightarrow$$

$$\Rightarrow\frac{1}{2z}-\frac{\pi}{2}\cot(\pi z)=\sum_{k=1}^{+\infty}\zeta(2k)z^{2k-1}\quad\text{per } |z|<1.$$

$$\zeta(2k)=1+O(1/2^{2k})\sim 1$$

$$\sqrt[k]{\zeta(2k)}\xrightarrow{k\to+\infty}1\Rightarrow\text{raggio di convergenza } 1.$$

Def.: si dicono numeri di Bernoulli i seguenti:

$$B_n=D^n\left(\frac{z}{e^z-1}\right)_{z=0}.$$

Raggio di convergenza $2\pi\Rightarrow\limsup\sqrt[n]{\frac{B_n}{n!}}=\frac{1}{2\pi}$.

Oss.: $f(z)=\frac{z}{e^z-1}+\frac{z}{2}=\frac{z+ze^z}{2(e^z-1)}=\frac{z(1+e^z)}{2(e^z-1)}$

$$f(-z)=\frac{-ze^z}{1-e^z}-\frac{z}{2}=\frac{-\cancel{2}ze^z-z+\cancel{ze^z}}{2(1-e^z)}=\frac{z(1+e^z)}{2(e^z-1)}\Rightarrow f \text{ pari}.$$

Allora $D^{2k-1}\left(\frac{z}{e^z-1}\right)_{z=0}=-D^{2k-1}\left(\frac{z}{2}\right)_{z=0}\begin{cases}-1/2 & k=1\\ 0 & k>1\end{cases}\Rightarrow$

$\Rightarrow B_1=-1/2,\ B_{2n-1}=0\ \forall n>1.$

Oss.: $1=\left(\sum_{n=0}^{+\infty}\frac{B_n}{n!}z^n\right)\frac{e^z-1}{z}=\left(\sum_{n=0}^{+\infty}\frac{B_n}{n!}z^n\right)\left(\sum_{m=1}^{+\infty}\frac{z^{m-1}}{m!}\right)=$

$$=\sum_{n=1}^{+\infty}\left(\sum_{k=0}^{n-1}\frac{B_k}{k!}\frac{n!}{(n-k)!}\right)\frac{z^{n-1}}{n!}=\sum_{n=1}^{+\infty}\left(\sum_{k=0}^{n-1}\binom{n}{k}B_k\right)\frac{z^{n-1}}{n!}$$

$B_0=1$, $\sum_{k=0}^{n-1}\binom{n}{k}B_k=0\ \forall n\geq 2\Rightarrow$

$\Rightarrow\sum_{k=0}^{n}\binom{n}{k}B_k=B_n$, quindi

$$B_{n-1}=-\frac{1+\binom{n}{1}B_1+\dots+\binom{n}{n-2}B_{n-2}}{\binom{n}{n-1}}\Rightarrow B_n\in\mathbb{Q}.$$

Oss.: $\cot(z)=i\frac{e^{iz}+e^{-iz}}{e^{iz}-e^{-iz}}=i\frac{e^{2iz}+1}{e^{2iz}-1}=i\left(1+\frac{2z}{z(e^{2iz}-1)}\right)=$

$$=i+\frac{1}{z}\cdot\frac{2iz}{e^{2iz}-1}=i+\frac{1}{z}\left(1-iz+\sum_{k=1}^{+\infty}\frac{(-1)^kB_{2k}(2z)^{2k}}{(2k)!}\right)=$$

$$=\frac{1}{z}+\sum_{k=1}^{+\infty}\frac{2(-1)^kB_{2k}}{(2k)!}(2z)^{2k-1}\Rightarrow$$

$$\Rightarrow\frac{1}{2z}-\frac{\pi}{2}\cot(\pi z)=\frac{1}{2}\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}z^{2k-1}}{(2k)!}$$

$$\| $$

$$\sum_{k=1}^{+\infty}\zeta(2k)z^{2k-1}\Rightarrow\zeta(2k)=\frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!}$$

$$\zeta(2k)=1+O\left(\frac{1}{4^k}\right)$$

$$(-1)^{k-1}B_{2k}=\frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k)$$

Oss.: $B_{2k}(-1)^{k-1}>0$. Inoltre, $\frac{2(2k)!}{(2\pi)^{2k}}\leq|B_{2k}|\leq\frac{2(2k)!\pi^2}{6(2\pi)^{2k}}\ \forall k\geq 1$

$$|B_{2(k+1)}|\geq\frac{2(2(k+1))!}{(2\pi)^{2(k+1)}}\overset{?}{\geq}\frac{2(2k)!\pi^2}{6(2\pi)^{2k}}$$

$$\frac{2(k+1)(2k+1)}{4\pi^2}\overset{?}{\geq}\frac{\pi^2}{6}$$

$$\pi^4\overset{?}{\leq}3(k+1)(2k+1),\ \text{vero per } k\geq 4.$$

Def.: si dice polinomio di Bernoulli n-esimo il seguente:

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}.$$

Fatto:
- $B_n(0)=B_n$
- $B_n(1)=\sum_{k=0}^{n}\binom{n}{k}B_k=(-1)^nB_n$

Oss.:

$$\frac{ze^{xz}}{e^z-1}=\left(\sum_{n=0}^{+\infty}\frac{B_n}{n!}z^n\right)\left(\sum_{m=0}^{+\infty}\frac{x^mz^m}{m!}\right)=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}\frac{B_k}{k!}\frac{x^{n-k}}{(n-k)!}\right)z^n=$$

$$=\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n}B_k\binom{n}{k}x^{n-k}\right)\frac{z^n}{n!}=\sum_{n=0}^{+\infty}\frac{B_n(x)}{n!}z^n.$$

Fatto:
- $B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}$;

per $y=1$, $B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x)$

- $B_n(x+1)-B_n(x)=nx^{n-1}$
- $\sum_{k=m}^{n-1}k^r=\frac{B_{r+1}(n)-B_{r+1}(m)}{r+1}$
- $B_n'(x)=nB_{n-1}(x)$

Teorema (formula di sommazione di Eulero): sia $f:[a,b]\to\mathbb{C}$ di classe C^1.
Allora
$$\sum_{a<k\leq b}f(k)=\int_a^bf(x)dx-\Big[B_1(\{x\})f(x)\Big]_a^b+\int_a^bB_1(\{x\})f'(x)dx.$$

Se $a=m$, $b=n$,

$$\sum_{m<k\leq n}f(k)=\int_m^nf(x)dx-B_1(f(n)-f(m))+\int_m^nB_1(\{x\})f'(x)dx.$$

Dim.: no. □

Prop.: $\exists$ il limite γ, con $0<\gamma<1$, della successione

$$\sum_{k=1}^{n}\frac{1}{k}-\log n.$$