Sommazione di Eulero $\Rightarrow \sum_{k=m}^{n} f(k)=\int_m^n f(x)dx + \frac{f(m)+f(n)}{2}+\int_m^n B_1(\{x\})f'(x)dx$

$(B_1(\{x\})=\{x\}-1/2)$

Abel: $\sum_{k=m}^{n} a_k f(k)=\left(\sum_{k=m}^{n} a_k\right)f(n)-\int_m^n\left(\sum_{m\le k\le \lfloor x\rfloor} a_k\right)f'(x)dx.$

Si usa per la costante di Eulero-Mascheroni γ.

$m=1$, $f(x)=1/x$. Formula di Eulero $\Rightarrow$

$$\Rightarrow \sum_{k=1}^{n}\frac{1}{k}=\int_1^n\frac{dx}{x}+\frac{1}{2}+\frac{1}{2n}-\int_1^n\frac{\{x\}-\frac{1}{2}}{x^2}dx=$$
$$=\log n+\frac{1}{2}+\frac{1}{2n}-\int_1^{+\infty}\frac{\{x\}-1/2}{x^2}dx+\int_n^{+\infty}\frac{\{x\}-1/2}{x^2}dx=$$
$$=\log n+\frac{1}{2}+\frac{1}{2n}-\int_1^{+\infty}\frac{\{x\}}{x^2}dx+\frac{1}{2}+O\left(\int_n^{+\infty}\frac{dx}{x^2}\right)=$$
$$=\log n+1-\int_1^{+\infty}\frac{\{x\}}{x^2}dx+O\left(\frac{1}{n}\right)=:\log n+\gamma+O\left(\frac{1}{n}\right).$$
$$0<\int_1^{+\infty}\frac{\{x\}}{x^2}dx<1 \Rightarrow \quad 0<\gamma<1 \Rightarrow$$
$$\Rightarrow \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k}-\log n=\gamma.$$

Oss.: se $f:[1,+\infty)\to\mathbb{R}$ è di classe C^1 infinitesima e non crescente, allora $\exists\, C>0$ t.c. $\sum_{1\le k\le x} f(k)=\int_1^x f(y)dy+C+O(f(x))$. Ex..

Oss.: si può dimostrare che $\sum_{k=1}^{n}\frac{1}{k}=\log n+\gamma+\frac{1}{2n}-\sum_{r=1}^{q}\frac{B_{2r}}{2r}\cdot\frac{1}{n^{2r}}+O\left(\frac{1}{n^{2q+2}}\right)$.

Def.: sia $\frac{1}{z\Gamma(z)}$ la funzione intera F di ordine 1 con zeri tutti semplici nei punti $-1,-2,-3,\dots$ e t.c. $\left(\frac{1}{z\Gamma(z)}\right)_{z=0}=1$ e $D\left(\frac{1}{z\Gamma(z)}\right)_{z=0}=\gamma$.

Deve essere $q(=\deg G)\le 1$, $\beta=1$, $\alpha=1$.

$G(z)=\log F(0)+\frac{F'}{F}(0)\,z=\gamma z \Rightarrow q=1.\ p=1.\ g=1.$

$$\frac{1}{z\Gamma(z)}=e^{\gamma z}\prod_{n=1}^{+\infty}\left(1+\frac{z}{n}\right)e^{-z/n}.$$

Prop.: vale la seguente formula (di Gauss):
$$\Gamma(z)=\lim_{n\to+\infty}\frac{n^z n!}{z(z+1)\dots(z+n)}. \qquad (z\ne -n,\ n\in\mathbb{N}\cup\{0\})$$

Dim.: $\frac{1}{\Gamma(z)}=z e^{\gamma z}\prod_{n=1}^{+\infty}\left(1+\frac{z}{n}\right)e^{-z/n}=z\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{\left(\gamma-\sum_{k=1}^{n}\frac{1}{k}\right)z}=$

$=z\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)\cdot e^{\lim_{n\to+\infty}\left(\gamma-\sum_{k=1}^{n}\frac{1}{k}\right)z}=z\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)n^{-z}\Rightarrow$

$\Rightarrow \Gamma(z)=\lim_{n\to+\infty}\frac{n^z\prod_{k=1}^{n}k}{z\prod_{k=1}^{n}(k+z)}=\lim_{n\to+\infty}\frac{n^z n!}{z(z+1)\dots(z+n)}.$ $\square$

Prop.: si ha $\operatorname{Res}_{z=-k}\Gamma(z)=\frac{(-1)^k}{k!}$.

Dim.: fissato k, sia $n\ge k$. Voglio $(\Gamma(z)(z+k))_{z=-k}$.

$\left(\frac{n^z n!\,(z+k)}{z(z+1)\dots(z+n)}\right)_{z=-k}=\frac{n^{-k}n!}{-k(-k+1)\dots(-2)\cdot(-1)\cdot1\cdot2\dots(n-k)}=\frac{(-1)^k}{k!}\left(\frac{n!}{(n-k)!\,n^k}\right)\xrightarrow{n\to+\infty}\frac{(-1)^k}{k!}.$ $\square$

Prop.: per $z\ne -n$, $n\in\mathbb{N}\cup\{0\}$ si ha la seguente formula (Eulero):
$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}.$$

Dim.: dalla dim. di Gauss, $n=\prod_{k=1}^{n-1}\frac{k+1}{k}\Rightarrow n^{-z}=\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^{-z}$

$\frac{1}{\Gamma(z)}=z\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)n^{-z}=z\lim_{n\to+\infty}\prod_{k=1}^{n-1}\left(1+\frac{1}{k}\right)^{-z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)=$

$=z\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^{-z}\left(1+\frac{z}{k}\right)\left(1+\frac{1}{n}\right)^z=z\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^{-z}\left(1+\frac{z}{n}\right)\Rightarrow$

$\Rightarrow \Gamma(z)=\frac{1}{z}\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}.$ $\square$

Prop.: si ha $\Gamma(z+1)=z\Gamma(z)$ per $z\in\mathbb{C}$.

Dim.: $\Gamma(z+1)=\frac{1}{z+1}\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{1}{n}\right)\left(1+\frac{z+1}{n}\right)^{-1}=$

$=\frac{1}{z+1}\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^z\left(\frac{n+1+z}{n}\cdot\frac{n}{n+1}\right)^{-1}=\frac{1}{z+1}\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n+1}\right)^{-1}=$

$=\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z\frac{1}{z+1}\prod_{k=1}^{n}\left(1+\frac{z}{k+1}\right)^{-1}=\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z\frac{1}{z+1}\prod_{k=2}^{n+1}\left(1+\frac{z}{k}\right)^{-1}=$

$=\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z\prod_{k=1}^{n+1}\left(1+\frac{z}{k}\right)^{-1}=\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z\left(1+\frac{z}{k}\right)^{-1}\frac{n+1}{n+1+z}=$

$=\lim_{n\to+\infty}\prod_{k=1}^{n}\left(1+\frac{1}{k}\right)^z\left(1+\frac{z}{k}\right)^{-1}=\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}=z\Gamma(z).$ $\square$

Oss.: $\Gamma(n+1)=n\Gamma(n)=n(n-1)\Gamma(n-1)=\dots=n(n-1)\cdot\ldots\cdot2\cdot\Gamma(1)=n!$.

Prop.: si ha la relazione
$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)} \qquad (z\in\mathbb{C}).$$

Dim.: $z\Gamma(z)\Gamma(1-z)=\Gamma(1+z)\Gamma(1-z)=\frac{1}{1-z^2}\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^{z+1}\left(1+\frac{1}{n}\right)^{1-z}\left(1+\frac{z+1}{n}\right)^{-1}\left(1+\frac{1-z}{n}\right)^{-1}=$

$=\frac{1}{1-z^2}\prod_{n=1}^{+\infty}\left(1+\frac{1}{n}\right)^2\left(\left(1+\frac{1}{n}\right)^2-\frac{z^2}{n^2}\right)^{-1}=\frac{1}{1-z^2}\prod_{n=1}^{+\infty}\left(\frac{\left(1+\frac{1}{n}\right)^2-\frac{z^2}{n^2}}{\left(1+\frac{1}{n}\right)^2}\right)^{-1}=$

$=\frac{1}{1-z^2}\prod_{n=1}^{+\infty}\left(1-\frac{z^2}{(n+1)^2}\right)^{-1}=\frac{1}{1-z^2}\prod_{n=2}^{+\infty}\left(1-\frac{z^2}{n^2}\right)^{-1}=\prod_{n=1}^{+\infty}\left(1-\frac{z^2}{n^2}\right)^{-1}=$

$=\frac{\pi z}{\sin(\pi z)}.$ $\square$

Formula di moltiplicazione di Gauss: $n\ge1$,
$$\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz).$$

Prop. (formula di duplicazione di Legendre):
$$\Gamma(z)\Gamma(z+1/2)=\sqrt{\pi}\,2^{1-2z}\Gamma(2z).$$

Oss.: $\Gamma(1/2)\cdot\Gamma(1)=\sqrt{\pi}\,\Gamma(2)\Rightarrow\Gamma(1/2)=\sqrt{\pi}$.

$\Gamma(3/2)=\frac{1}{2}\Gamma(1/2)=\frac{\sqrt{\pi}}{2}<1.$

Oss.: $\boxed{\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)}{\Gamma(z/2)}}=\frac{\Gamma\left(\frac{1}{2}-\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}\right)}{\Gamma\left(\frac{z}{2}\right)\Gamma\left(1-\frac{z}{2}\right)}=\frac{\sin(\pi z/2)}{\pi}\sqrt{\pi}\,2^{1-(1-z)}\Gamma(1-z)=$

$\boxed{=\frac{\sin(\pi z/2)}{\sqrt{\pi}}2^z\Gamma(1-z).}$