Dim. (di duplicazione di Legendre): $\Gamma(z)=\lim_{n\to+\infty}\dfrac{n^z n!}{z(z+1)\dots(z+n)}=$

$$=\lim_{n\to+\infty}\frac{n^z(n-1)!}{z(z+1)\dots(z+n-1)}\cdot\frac{n}{z+n}=\lim_{n\to+\infty}\frac{n^z(n-1)!}{z(z+1)\dots(z+n-1)}$$

$$\frac{2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)}=\lim_{n\to+\infty}\frac{2^{2z-1}}{\frac{(2n)^{2z}(2n-1)!}{2z(2z+1)\dots(2z+2n-1)}}\cdot\frac{n^z(n-1)!}{z(z+1)\dots(z+n-1)}\cdot\frac{n^{z+\frac{1}{2}}(n-1)!}{(z+\frac{1}{2})(z+3/2)\dots(z+n-1/2)}=$$

$$=\lim_{n\to+\infty}\frac{2^{2z-1}n^{2z+\frac{1}{2}}((n-1)!)^2\,2^n\,(2z)(2z+1)\dots(2z+2n-1)}{(2n)^{2z}(2n-1)!\,z(z+1)\dots(z+n-1)(2z+1)(2z+3)\dots(2z+2n-1)}=$$

$$=\lim_{n\to+\infty}\frac{2^{2n-1}n^{1/2}((n-1)!)^2}{(2n-1)!}=\text{costante}.$$ Per la prop. precedente,

$$\Gamma^2(1/2)=\frac{\pi}{\sin(\pi/2)}=\pi\Rightarrow\Gamma(1/2)=\sqrt{\pi}\Rightarrow\text{costante}=\sqrt{\pi}.\ \square$$

Prop.: si ha $\dfrac{\Gamma'}{\Gamma}(z)=-\gamma-\dfrac{1}{z}+\sum_{n=1}^{+\infty}\dfrac{z}{n(n+z)}$ (fuori dai poli di Γ).

Oss.: prop. $\Rightarrow\Gamma'(1)=-\gamma-1+\sum_{n=1}^{+\infty}\dfrac{1}{n(n+1)}=-\gamma$.

Dim. (della prop.): $\Gamma(z)=\dfrac{1}{z}e^{-\gamma z}\prod_{n=1}^{+\infty}\left(1+\dfrac{z}{n}\right)^{-1}e^{\frac{z}{n}}$

$$\log(\Gamma(z))=-\log z-\gamma z+\sum_{n=1}^{+\infty}\left(\frac{z}{n}-\log\left(1+\frac{z}{n}\right)\right)$$

$$\frac{\Gamma'}{\Gamma}(z)=-\gamma-\frac{1}{z}+\sum_{n=1}^{+\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)=-\gamma-\frac{1}{z}+\sum_{n=1}^{+\infty}\frac{z}{n(n+z)}.\ \square$$

$\int_0^{+\infty}e^{-x}x^{z-1}dx$ converge per $\operatorname{Re}z>0$. Definisce una funzione olo..

Teorema: per $\operatorname{Re}z>0$ si ha $\Gamma(z)=\int_0^{+\infty}e^{-x}x^{z-1}dx$.

Dim.: 1) $\lim_{n\to+\infty}\int_0^n\left(1-\frac{x}{n}\right)^n x^{z-1}dx=\int_0^{+\infty}e^{-x}x^{z-1}dx$

2) $\int_0^n\left(1-\frac{x}{n}\right)^n x^{z-1}dx=\dfrac{n^z n!}{z(z+1)\dots(z+n)}$.

1) $0\leq e^{-x}-\left(1-\frac{x}{n}\right)^n\leq\frac{x^2}{n}e^{-x}$ $(*)$

$$1+\frac{x}{n}\leq1+\frac{x}{n}+\frac{x^2}{2!n^2}+\frac{x^3}{3!n^3}+\dots\leq1+\frac{x}{n}+\frac{x^2}{n^2}+\dots\Rightarrow$$

$$\Leftrightarrow\ 1+\frac{x}{n}\leq e^{x/n}\leq\frac{1}{1-x/n}\ \Rightarrow$$

$$\Rightarrow\left(1+\frac{x}{n}\right)^n\leq e^x,\ \left(1-\frac{x}{n}\right)^n\leq e^{-x}\ \Rightarrow$$

$$\Rightarrow e^{-x}-\left(1-\frac{x}{n}\right)^n=e^{-x}\left(1-e^x\left(1-\frac{x}{n}\right)^n\right)\leq e^{-x}\left(1-\left(1-\frac{x^2}{n^2}\right)^n\right).$$

Bernoulli: $\alpha\geq0$, $(1-\alpha)^n\geq1-n\alpha\Rightarrow$

$$\Rightarrow\ e^{-x}(1-(1-x^2/n^2)^n)\leq e^{-x}\left(1-\left(1-\frac{x^2}{n}\right)\right)=\frac{x^2}{n}e^{-x}.$$

$$\int_0^n\left(1-\frac{x}{n}\right)^n x^{z-1}dx=\int_0^n e^{-x}x^{z-1}dx-\left(\int_0^n e^{-x}x^{z-1}dx-\int_0^n\left(1-\frac{x}{n}\right)^n x^{z-1}dx\right)$$

$$\left|\int_0^n\left(e^{-x}-\left(1-\frac{x}{n}\right)^n\right)x^{z-1}dx\right|\leq\int_0^n x^{\operatorname{Re}z-1}\left(e^{-x}-(1-x/n)^n\right)dx\leq$$

$$\leq\frac{1}{n}\int_0^n x^{\operatorname{Re}z-1+2}e^{-x}dx\ll\frac{1}{n}\Rightarrow\lim_{n\to+\infty}\int_0^n\left(1-\frac{x}{n}\right)^n x^{z-1}dx=\int_0^{+\infty}e^{-x}x^{z-1}dx.$$

2) $$\int_0^n\left(1-\frac{x}{n}\right)^n x^{z-1}dx\overset{\text{c.d.v.}}{=}n^z\int_0^1(1-y)^n y^{z-1}dy$$

$$\int_0^1(1-y)^n y^{z-1}dy\underset{\text{per parti}}{=}\frac{n}{z}\int_0^1(1-y)^{n-1}y^z dy=\dots\text{itero}\dots=$$

$$=\frac{n(n-1)\dots2\cdot1}{z(z+1)\dots(z+n-1)}\int_0^1 y^{z+n-1}dy=\frac{n!}{z(z+1)\dots(z+n)}.\ \square$$

Teorema (Stirling): sia $\varepsilon>0$. Si ha la seguente formula asintotica

$$\log(\Gamma(z))=\left(z-\frac{1}{2}\right)\log z-z+\log\sqrt{2\pi}+O\left(\frac{1}{|z|}\right)$$

uniformemente per $|z|\geq\varepsilon$ e $|\arg z|\leq\pi-\varepsilon$.

In particolare, $\log n!=(n+1/2)\left(\log n+\frac{1}{n}+O\left(\frac{1}{n^2}\right)\right)-n-1+\log\sqrt{2\pi}+O\left(\frac{1}{|n+1|}\right)=$
$=(n+1/2)\log n-n+\log\sqrt{2\pi}+O\left(\frac{1}{n}\right)$.

Dim.: $\log(\Gamma(z))=\lim_{n\to+\infty}\log\left(\dfrac{n^z n!}{z(z+1)\dots(z+n)}\right)$

$$\log\left(\frac{n^z n!}{z(z+1)\dots(z+n)}\right)=z\log n-\log(z+n)-\sum_{k=1}^n\log\left(1+\frac{z-1}{k}\right).$$

$f_z(x)=\log\left(1+\frac{z-1}{x}\right)$, Eulero-McLaurin $\Rightarrow$

$$\Rightarrow\sum_{k=1}^n\log\left(1+\frac{z-1}{k}\right)=\int_1^n\log\left(1+\frac{z-1}{x}\right)dx+\frac{1}{2}\left(\log z+\log(z+n-1)-\log n\right)+$$
$$+\int_1^n B_1(\{x\})\left(\frac{1}{z+x-1}-\frac{1}{x}\right)dx\Rightarrow$$

$$\Rightarrow\log\left(\frac{n^z n!}{z(z+1)\dots(z+n)}\right)=\dots\text{conti}\dots=\left(z-\frac{1}{2}\right)\log z-\left(z+n+\frac{1}{2}\right)\log\left(1+\frac{z-1}{n}\right)+\log\left(1-\frac{1}{z+n}\right)+$$
$$-\int_1^n B_1(\{x\})\left(\frac{1}{z+x-1}-\frac{1}{x}\right)dx$$

$$\left(z+n+\frac{1}{2}\right)\log\left(1+\frac{z-1}{n}\right)\sim\frac{z-1}{n}\left(z+n+\frac{1}{2}\right)\xrightarrow{n\to+\infty}z-1\ \left(\text{con un }O(1/n)\right)$$

$$\log\left(\frac{n^z n!}{z(z+1)\dots(z+n)}\right)=\left(z-\frac{1}{2}\right)\log z-z+1+O\left(\frac{1}{n}\right)-\int_1^{+\infty}B_1(\{x\})\left(\frac{1}{z+x-1}-\frac{1}{x}\right)dx+$$
$$+\int_n^{+\infty}B_1(\{x\})\left(\frac{1}{z+x-1}-\frac{1}{x}\right)dx=$$

$=\left(z-\frac{1}{2}\right)\log z-z+C+O\left(\frac{1}{n}\right)+R_z(n)$. Ricordiamo che $DB_k(x)=kB_{k-1}(x)$

$\{x\}-\frac{1}{2}=x-\lfloor x\rfloor-\frac{1}{2}$

$\int_1^n\left(x-\lfloor x\rfloor-\frac{1}{2}\right)dx$, $\underbrace{\dfrac{(x-\lfloor x\rfloor)^2}{2}-\dfrac{x-\lfloor x\rfloor}{2}+\dfrac{1}{12}}_{\frac{1}{2}B_2(\{x\})}$ è una primitiva