(fine dim. di Stirling):

$$\int_1^{+\infty} \frac{B_1(\{x\})}{z+x-1}\,dx \overset{\text{per parti}}{=} \left[\frac{B_2(\{x\})}{2(z+x-1)}\right]_1^{+\infty} + \int_1^{+\infty} \frac{B_2(\{x\})}{2(z+x-1)^2}\,dx =$$

$$= -\frac{1}{12z} + \int_1^{+\infty} \frac{B_2(\{x\})}{2(z+x-1)^2}\,dx$$

$$C = 1 + \int_1^{+\infty} \frac{B_1(\{x\})}{x}\,dx$$

$$-\frac{1}{2}\int_1^{+\infty} \frac{B_2(\{x\})}{(z+x-1)^2}\,dx = -\frac{1}{2}\int_0^{+\infty} \frac{B_2(\{x\})}{(z+x)^2}\,dx \overset{?}{\ll_\varepsilon} \frac{1}{|z|}$$

$$\left|\int \cdots\right| \underset{\text{costanti}<1}{\leqslant} \int_0^{+\infty} \frac{dx}{|z+x|^2} \leqslant$$

$|\arg z| \leqslant \pi - \varepsilon$

$$|z+x|^2 = |z|^2 + x^2 + 2\langle z, x\rangle \geqslant |z|^2 + x^2 - 2|z|x\cos\varepsilon$$

$$x - |z|\cos\varepsilon = y$$

$$\leqslant \int_{-|z|\cos\varepsilon}^{+\infty} \frac{dy}{|z|^2\sin^2\varepsilon + y^2} \leqslant \int_{-\infty}^{+\infty} \frac{dy}{|z|^2\sin^2\varepsilon + y^2} =$$

c'è un c.d.v.

$$= \frac{1}{|z|\sin\varepsilon}\int_{-\infty}^{+\infty} \frac{dy}{y^2+1} \ll_\varepsilon \frac{1}{|z|}.$$

$$\Gamma(z)\Gamma(z+1/2) = \sqrt{\pi}\,2^{1-2z}\,\Gamma(2z) \Longrightarrow$$

$$\Longrightarrow \log\Gamma(z) + \log\Gamma(z+1/2) = \frac{1}{2}\log\pi + (1-2z)\log 2 + \log\Gamma(2z)$$; ho anche

$$\log\Gamma(z) = (z-1/2)\log z - z + C + \mathcal{O}_\varepsilon\left(\frac{1}{|z|}\right) \Longrightarrow$$

$$z\log z - z\log\left(1+\frac{1}{2z}\right) = z\log z + \frac{1}{2} + \mathcal{O}\left(\frac{1}{|z|}\right)$$

$$\Longrightarrow (z-1/2)\log z - \cancel{z} + C + \overbrace{z\log(z+1/2)} - \cancel{z} - 1/2 + \cancel{C} - \frac{1}{2}\log\pi + (2z-1)\log 2 +$$

$$-(2z-1/2)\log(2z) + \cancel{2z} - \cancel{C} = \mathcal{O}_\varepsilon\left(\frac{1}{|z|}\right) \Longrightarrow$$

$$\Longrightarrow -\frac{1}{2}\log z + C - \frac{1}{2}\log\pi - \frac{1}{2}\log 2 + \frac{1}{2}\log z = C - \frac{1}{2}\log(2\pi) \ll_\varepsilon \frac{1}{|z|} \Longrightarrow$$

$$\Longrightarrow C = \frac{1}{2}\log(2\pi). \qquad \square$$

Cor.: se $|z| \geqslant \varepsilon$ e $|\arg z| \leqslant \pi - \varepsilon$ si ha

$$\frac{\Gamma'}{\Gamma}(z) = \log z + \mathcal{O}_\varepsilon\left(\frac{1}{|z|}\right).$$

Dim.: $$f'(z_0) = -\frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{(z-z_0)^2}\,dz \ll_\varepsilon \frac{1}{|z_0|}$$

γ: cerchio di raggio $\varepsilon/2$

$$f(z) = \log\Gamma(z) - (z-1/2)\log z + z - \frac{1}{2}\log(2\pi) \ll_\varepsilon \frac{1}{|z|} \ll \frac{1}{|z_0|}$$

$$f'(z_0) = \frac{\Gamma'}{\Gamma}(z_0) - \log z_0 \ll_\varepsilon \frac{1}{|z_0|}. \qquad \square$$

Cor.: per $\varepsilon > 0$ e $|z| \geqslant \varepsilon$, $|\arg z| \leqslant \pi - \varepsilon$ si ha

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left(1 + \mathcal{O}_\varepsilon\left(\frac{1}{|z|}\right)\right).$$

In particolare, $\Gamma(n+1) = n\Gamma(n) \Longrightarrow n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$.

Cor.: se $k \geqslant 1$, allora $(-1)^k B_{2k} = 4\sqrt{\pi k}\left(\frac{k}{e\pi}\right)^{2k}\left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right)$.

Dim.: si ha infatti $(-1)^k B_{2k} = \frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k)$, ma

$$\zeta(2k) = \sum_{n=1}^{+\infty}\frac{1}{n^{2k}} \leqslant 1 + \int_1^{+\infty}\frac{dx}{x^{2k}} = 1 + \frac{1}{2k-1} = 1 + \mathcal{O}\left(\frac{1}{k}\right). \qquad \square$$

Si ritrova il raggio di convergenza di $\frac{z}{e^z-1}$.

Cor.: sia $z = x+iy$, $x_1 \leqslant x \leqslant x_2$. Allora

$$|\Gamma(x+iy)| = \sqrt{2\pi}\,|y|^{x-\frac{1}{2}}\,e^{-\frac{\pi}{2}|y|}\left(1 + \mathcal{O}\left(\frac{1}{|y|}\right)\right).$$

Dim.: $$\log\Gamma(x+iy) = \left(x - \frac{1}{2} + iy\right)\log(x+iy) - x - iy + \frac{1}{2}\log(2\pi) + \mathcal{O}\left(\frac{1}{|y|}\right)$$

$$\mathrm{Re}\left(\log\Gamma(x+iy)\right) = \mathrm{Re}\left(\left(x-\frac{1}{2}+iy\right)\left(\log(iy) + \log\left(1 - \frac{ix}{y}\right)\right) - x + \frac{1}{2}\log(2\pi)\right) + \mathcal{O}\left(\frac{1}{|y|}\right) =$$

$$= (x-1/2)\log|y| + \underbrace{iy\left(\frac{\pi}{2}i\,\mathrm{sgn}(y)\right)}_{-|y|\frac{\pi}{2}} - x + \frac{1}{2}\log(2\pi) + iy\left(-\frac{ix}{y}\right) + \mathcal{O}\left(\frac{1}{|y|}\right) =$$

= TORNA. $\square$

Def.: sia $f: \mathbb{R} \to \mathbb{C}$; si dice che f tende rapidamente a 0 per $|x| \to +\infty$ se $\lim_{x\to\pm\infty} |x|^n f(x) = 0$ $\forall n \in \mathbb{N}\cup\{0\}$.

Oss.: rapida tendenza a 0 $\Longleftrightarrow$ limitatezza $f(x)|x|^n$ $\forall n \in \mathbb{N}\cup\{0\}$.

Def.: si dice spazio di Schwarz $\mathcal{S}$ lo spazio su $\mathbb{C}$ delle funzioni $f \in C^\infty(\mathbb{R})$ (a valori complessi) tendenti rapidamente a 0 insieme a tutte le loro derivate.

Oss.: $D^k: \mathcal{S} \to \mathcal{S}$ $\forall k \geqslant 0$.

Notazione: indichiamo con M^k l'operatore $(M^k f)(x) = x^k f(x) \Longrightarrow$
$\Longrightarrow M^k: \mathcal{S} \to \mathcal{S}$ $\forall k \geqslant 0$.

Consideriamo la trasformata di Fourier in $\mathcal{S}$ definita come

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i\xi x}\,dx.$$ È ben definita.

Oss.: $|\hat{f}(\xi)| \leqslant \int_{\mathbb{R}}|f(x)|\,dx < +\infty \Longrightarrow \hat{f}$ è limitata se $f \in \mathcal{S}$.

Lemma: $\hat{\ }: \mathcal{S} \to \mathcal{S}$.

Dim.: $$\hat{f}'(\xi) = -2\pi i\int_{-\infty}^{+\infty} x f(x)e^{-2\pi i\xi x}\,dx = -2\pi i\,\widehat{Mf}(\xi) \Longrightarrow$$

$$\Longrightarrow D\hat{f} = (-2\pi i)\widehat{Mf} \Longrightarrow D^k\hat{f} = (-2\pi i)^k\widehat{M^k f} \Longrightarrow \hat{f} \in C^\infty(\mathbb{R}).$$

$M^k\hat{f} = \left(\frac{1}{2\pi i}\right)^k\widehat{D^k f}$. Infatti,

$$\xi\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\,\xi\,e^{-2\pi i\xi x}\,dx \overset{\text{parti}}{=} \left[-\frac{1}{2\pi i}e^{-2\pi i\xi x}f(x)\right]_{-\infty}^{+\infty} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} f'(x)e^{-2\pi i\xi x}\,dx =$$

$$= \frac{1}{2\pi i}\widehat{Df}(\xi).$$

Sfruttando l'oss. di limitatezza di $\hat{f}$ e l'oss. di caratterizzazione di rapida tendenza a 0, con un po' di giri di parole si ha la tesi. $\square$