Lemma (formula di Poisson): se $f \in \mathcal{S}$ allora
$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

Dim.: sia $g(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. g ha periodo 1.

$f \in \mathcal{S} \Rightarrow \sum_{n\in\mathbb{Z}} D^k f(x+n)$ converge uniformemente $\Rightarrow$ è $D^k g$,
quindi $g \in C^\infty(\mathbb{R})$.
$$g(x) = \sum_{m\in\mathbb{Z}} c_m e^{2\pi i m x}, \quad c_m = \int_0^1 g(x) e^{-2\pi i m x}\,dx =$$
$$= \int_0^1 \sum_{n\in\mathbb{Z}} f(x+n) e^{-2\pi i m x}\,dx = \sum_{n\in\mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m x}\,dx =$$
$$\underset{y=x+n}{=} \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(y) e^{-2\pi i m(y-n)}\,dy = \int_{\mathbb{R}} f(x) e^{-2\pi i m x}\,dx = \hat{f}(m).$$
Basta quindi guardare $g(0)$. $\square$

Lemma: sia $f(x) = e^{-\pi x^2}$ $(x \in \mathbb{R})$. Allora $f \in \mathcal{S}$ e inoltre $\hat{f} = f$.

Dim.: $f \in \mathcal{S}$ é facile.
$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2 - 2\pi i \xi x}\,dx \Rightarrow$$
$$\Rightarrow D\hat{f}(\xi) = -2\pi i \int_{\mathbb{R}} x e^{-\pi x^2 - 2\pi i \xi x}\,dx \overset{\text{parti}}{=}$$
$$= \left[i e^{-\pi x^2 - 2\pi i \xi x} \right]_{-\infty}^{+\infty} + i(2\pi i \xi) \int_{\mathbb{R}} e^{-\pi x^2 - 2\pi i \xi x}\,dx =$$
$$= -2\pi \xi \hat{f}(\xi).$$
Abbiamo $u'(\xi) = -2\pi \xi u(\xi) \Rightarrow \frac{u'}{u}(\xi) = -2\pi\xi \Rightarrow$
$$\Rightarrow \log u(\xi) = -\pi\xi^2 + c \Rightarrow u(\xi) = Ce^{-\pi\xi^2} \Rightarrow$$
$$\Rightarrow \hat{f}(\xi) = Ce^{-\pi\xi^2}.$$
$$\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1 \Rightarrow C = 1. \quad \square$$

Oss.: la serie $\sum_{n\in\mathbb{Z}} e^{-\pi n^2 z}$ $(\operatorname{Re} z > 0)$ converge totalmente per $\operatorname{Re} z \geq \varepsilon > 0$.

Def.: sia $z = x+iy$ con $x > 0$. Si dice funzione ϑ di Jacobi la seguente serie totalmente convergente:
$$\vartheta(z) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 z}.$$

Lemma: per $x = \operatorname{Re} z > 0$ si ha
$$\vartheta(z) = \frac{1}{\sqrt{z}} \vartheta\left(\frac{1}{z}\right). \qquad (*)$$

Dim.: possiamo dimostrare $(*)$ per $z = x > 0$, il resto segue per prolungamento analitico.
Sia $f(\xi) = e^{-\pi\xi^2}$ e sia $f_x(\xi) = f(\sqrt{x}\,\xi) = e^{-\pi x \xi^2}$.
$$\hat{f}_x(\xi) = \int_{\mathbb{R}} f(\sqrt{x}\,t) e^{-2\pi i \xi t}\,dt \underset{s = t\sqrt{x}}{=} \frac{1}{\sqrt{x}} \int_{\mathbb{R}} f(s) e^{-2\pi i \frac{\xi}{\sqrt{x}} s}\,ds =$$
$$= \frac{1}{\sqrt{x}} \hat{f}\left(\frac{\xi}{\sqrt{x}}\right) = \frac{1}{\sqrt{x}} f\left(\frac{\xi}{\sqrt{x}}\right) = \frac{1}{\sqrt{x}} f_{\frac{1}{x}}(\xi).$$
$$\sum_{n\in\mathbb{Z}} f_x(n) \underset{\substack{\text{lemma}\\\text{Poisson}}}{=} \sum_{n\in\mathbb{Z}} \hat{f}_x(n) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{x}} f_{\frac{1}{x}}(n), \text{ cioè}$$
$$\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}} \sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{x} n^2} = \frac{1}{\sqrt{x}} \vartheta\left(\frac{1}{x}\right). \quad \square$$

Teorema (Riemann): la funzione $\zeta(s)$ ($s = \sigma + it$) è meromorfa in $\mathbb{C}$ con un polo semplice in $s=1$ con residuo 1. Inoltre, posto
$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$, allora ξ è intera e fornisce il prolungamento analitico di ζ e si ha
$$\xi(s) = \xi(1-s).$$

Dim.: $\sigma > 0 \Rightarrow \Gamma\left(\frac{s}{2}\right) = \int_0^{+\infty} e^{-x} x^{\frac{s}{2}} \frac{dx}{x} \Rightarrow$
$$\Rightarrow \frac{\pi^{-s/2}}{n^s} \Gamma\left(\frac{s}{2}\right) = \int_0^{+\infty} e^{-x} \left(\frac{x}{\pi n^2}\right)^{s/2} \frac{dx}{x}.$$
$$\sigma > 1 \Rightarrow \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{-x} \left(\frac{x}{\pi n^2}\right)^{s/2} \frac{dx}{x} \underset{\substack{y = \frac{x}{\pi n^2}\\ \frac{dx}{x} = \frac{dy}{y}}}{=}$$
$$= \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y} = \int_0^{+\infty} \sum_{n=1}^{+\infty} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y} =$$
$$= \frac{1}{2} \int_0^{+\infty} (\vartheta(y) - 1) y^{s/2} \frac{dy}{y} = \frac{1}{2} \left(\int_0^1 + \int_1^{+\infty} \right) (\vartheta(y) - 1) y^{s/2} \frac{dy}{y}$$
$$\frac{1}{2} \int_0^1 (\vartheta(y) - 1) y^{s/2} \frac{dy}{y} \underset{\substack{x = \frac{1}{y}\\ \frac{dy}{y} = -\frac{dx}{x}}}{=} \frac{1}{2} \int_1^{+\infty} \left(\vartheta\left(\frac{1}{x}\right) - 1\right) x^{-\frac{s}{2}} \frac{dx}{x} =$$
$$= \frac{1}{2} \int_1^{+\infty} (\vartheta(x) - 1) \sqrt{x} \cdot x^{-s/2} \frac{dx}{x} + \frac{1}{2} \int_1^{+\infty} x^{\frac{1-s}{2}} \frac{dx}{x} - \frac{1}{2} \int_1^{+\infty} x^{-\frac{s}{2}} \frac{dx}{x} =$$
$$= \frac{1}{2} \int_1^{+\infty} (\vartheta(x) - 1) x^{\frac{1-s}{2}} \frac{dx}{x} + \frac{1}{s-1} - \frac{1}{s} = \frac{1}{2} \int_1^{+\infty} (\vartheta(x) - 1) x^{\frac{1-s}{2}} \frac{dx}{x} + \frac{1}{s(s-1)}.$$
$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \frac{1}{2} \int_1^{+\infty} (\vartheta(x) - 1)\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) \frac{dx}{x} \Rightarrow$$
$$\Rightarrow \xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} + \frac{s(s-1)}{4} \int_1^{+\infty} (\vartheta(x) - 1)\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right) \frac{dx}{x}$$
$\Rightarrow$ ξ è intera e ovviamente $\xi(1-s) = \xi(s)$.

stimando $\frac{1}{2}(\vartheta(x) - 1)$
$$\zeta(s) = \xi(s) \frac{\pi^{s/2}}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)} \frac{1}{s-1} \quad \text{e}$$
$$\operatorname*{Res}_{s=1} \zeta = \lim_{s\to 1} (\zeta(s)(s-1)) = \xi(1) \frac{\pi^{1/2}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{1}{2} \frac{\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} = 1. \quad \square$$

Oss.: ξ non ha zeri fuori dalla striscia $0 \leq \sigma \leq 1$.

Cor.: $\zeta(-2k) = 0 \quad \forall\, k \geq 1$.

Oss.: all'interno della striscia $0 \leq \sigma \leq 1$, ζ e ξ si annullano insieme.
ρ zero $\Rightarrow$ $1-\rho$ zero. ξ reale sui reali, quindi
ρ zero $\Rightarrow$ $\bar{\rho}$ zero.

Oss.: definendo $\Xi(s) = \xi(is + 1/2)$, $\Xi(-s) = \xi(1/2 - is) =$
$= \xi(1/2 + is) = \Xi(s)$.

Se $s = x \in \mathbb{R}$, $\overline{\xi\left(\frac{1}{2} + ix\right)} = \xi\left(\frac{1}{2} - ix\right) = \xi(1/2 + ix)$, $\Xi(x) = \Xi(-x) \Rightarrow$
$\Rightarrow$ $\Xi(x)$ é reale.

Oss.: $\zeta(s) \neq 0$ per $\sigma > 1$. Infatti
$$\left|\zeta(s) \prod_{p\leq N} \left(1 - \frac{1}{p^s}\right)\right| = \left|\prod_{p>N} \left(1 - \frac{1}{p^s}\right)^{-1}\right| = \left|1 + \sum_{\substack{p\mid n \Rightarrow\\ \Rightarrow p>N}} \frac{1}{n^s}\right| \geq$$
$$\geq 1 - \sum_{n>N} \frac{1}{n^\sigma} = 1 + O\left(\frac{1}{N^{\sigma-1}}\right).$$

Oss.: $0 \leq \sigma \leq 1$, $t = 0$.
$$\xi(\sigma) = \frac{1}{2} + \frac{\sigma(\sigma-1)}{4} \int_1^{+\infty} (\vartheta(x) - 1)\left(x^{\frac{\sigma}{2}} + x^{\frac{1-\sigma}{2}}\right) \frac{dx}{x}$$
$$\frac{\vartheta(x) - 1}{2} = \sum_{n=1}^{+\infty} e^{-\pi n^2 x} \leq \sum_{n=1}^{+\infty} e^{-\pi n x} = \frac{1}{e^{\pi x} - 1} \leq \frac{1}{2\sqrt{x}} \Rightarrow$$
$$e^{\pi x} \geq 1 + \pi x \Rightarrow e^{\pi x} - 1 \geq \pi x \underset{x \geq 1 \Rightarrow x \geq \sqrt{x}}{\geq} 2\sqrt{x}$$
$$\Rightarrow \int_1^{+\infty} \left(x^{\frac{\sigma-1}{2}} + x^{-\frac{\sigma}{2}}\right) \frac{dx}{x} = \frac{2}{\sigma(1-\sigma)}.$$
$$\xi(\sigma) \geq \frac{1}{2} - \frac{\sigma(1-\sigma)}{4} \frac{2}{\sigma(1-\sigma)} = 0.$$

Oss.: $\xi(0) = 1/2$, $\zeta(s) = \frac{\pi^{s/2}}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)} \underbrace{\frac{1}{s-1}}_{-\frac{1}{2}} \xi(s)$, in 0: $\zeta(0) = -\frac{1}{2}$.
$\to 1$