$$\xi(s)=(s-1)\Gamma\left(\tfrac{s}{2}+1\right)\pi^{-\frac{s}{2}}\zeta(s)$$

$$s=2n \Rightarrow \Gamma\left(\tfrac{s}{2}+1\right)=\Gamma(n+1)=n! \sim e^{(n+\frac{1}{2})\log n-n+(\cdots)} \ll_\varepsilon e^{n^{1+\varepsilon}} \Rightarrow$$

$\Rightarrow$ per $\sigma>1$ abbiamo $\operatorname{ord}\xi\geq 1$ (e se fosse $=$, non è un min).

$\zeta(s), \pi^{-s/2}$ lim., $s-1$ "piccola"

Per simmetria, ci manca solo $\frac{1}{2}\leq\sigma\leq 1$.

Se fosse proprio ordine 1, avremmo

$\xi(s)=e^{as+b}\prod_\rho(\cdots)$. Se non ci fossero zeri, $\prod_\rho=1 \Rightarrow \xi(s) \ll e^{a|s|}$, assurdo.

Lemma: per $\sigma\geq\varepsilon$ si ha $\zeta(s)\ll_\varepsilon |s|$ ($|t|$) uniformemente.

Dim.: $\sigma>1$
$$\sum_{n\leq x}\frac{1}{n^s} \underset{\substack{\downarrow\\ \text{sommazione}\\ \text{parziale}}}{=} \frac{\lfloor x\rfloor}{x^s}+s\int_1^x\frac{\lfloor u\rfloor}{u^{s+1}}\,du=\frac{\lfloor x\rfloor}{x^s}+s\int_1^s\frac{du}{u^s}-s\int_1^s\frac{\{u\}}{u^{s+1}}\,du=$$
$$=\frac{\lfloor x\rfloor}{x^s}+\frac{s}{s-1}-s\int_1^x\frac{\{u\}}{u^{s+1}}\,du \Rightarrow \zeta(s)=\frac{1}{s-1}+1-s\int_1^{+\infty}\frac{\{u\}}{u^{s+1}}\,du;$$

è l'estensione a $\sigma>0$ (l'integrale converge unif. per $\sigma\geq\varepsilon$).

$\ulcorner s=1 \Rightarrow \displaystyle\sum_{n\leq x}\frac{1}{n}=1-\frac{\{x\}}{x}+\log x-\int_1^x\frac{\{u\}}{u^2}\,du \Rightarrow$

$$\Rightarrow \lim_{x\to+\infty}\left(\sum_{n\leq x}\frac{1}{n}-\log x\right)=1-\int_1^{+\infty}\frac{\{u\}}{u^2}\,du=\gamma. \lrcorner$$

Con i pol. di Bernoulli,
$$\zeta(s)=\frac{1}{s-1}+1-\frac{s}{2}\int_1^{+\infty}\frac{du}{u^{s+1}}-s\int_1^{+\infty}\frac{B_1(\{u\})}{u^{s+1}}\,du=$$
$$=\frac{1}{s-1}+1-\frac{1}{2}-s\int_1^{+\infty}\frac{B_1(\{u\})}{u^{s+1}}\,du$$

integrando per parti, si ottiene il prolungamento fin dove si vuole

$$\int B_1(\{u\})\,du=\frac{1}{2}B_2(\{u\})+c$$

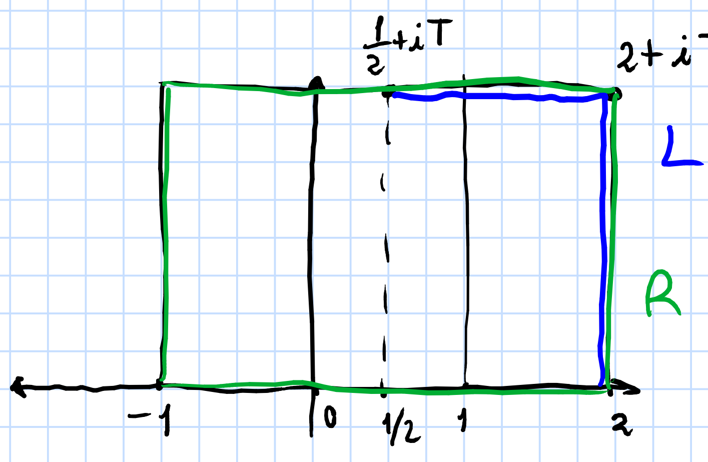
$$|\zeta(s)|\ll 1+|s|\int_1^{+\infty}\frac{du}{\underbrace{u^{1+\varepsilon}}_{\sigma\geq\varepsilon}} \ll \frac{|s|}{\varepsilon}\ll_\varepsilon |s|. \quad\square$$

Curiosità: congettura di Lindelof.

Prop. (formula di Riemann-Von Mangoldt):
detto $N(T)=\#\{\rho=\beta+i\gamma \mid \zeta(\rho)=0,\ 0\leq\beta\leq 1,\ 0<\gamma<T\}$, si ha
$$N(T)=\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}+\underbrace{O(\log T)}_{R}\quad(T\to+\infty).$$

$\ulcorner LH \Rightarrow R=o(\log T)$, $RH \Rightarrow R=O\left(\dfrac{\log T}{\log\log T}\right). \lrcorner$

Dim.:



$$N(T)=\frac{1}{2\pi}\Delta_R\arg\xi(s)$$
$$\xi(s)=\xi(1-s)=\overline{\xi(1-\bar{s})}$$

Per motivi di coniugio, $N(T)=\frac{1}{\pi}\Delta_L\arg\xi(s)$ (tra 0 e 1 non conta perché lì è reale positiva).

$$\xi(s)=\frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s)$$
$$\frac{1}{\pi}\Delta_L\arg\xi(s)=\frac{1}{\pi}\Delta_L\arg(s-1)+\frac{1}{\pi}\Delta_L\arg\pi^{-s/2}+$$
$$+\frac{1}{\pi}\Delta_L\arg\Gamma\left(\tfrac{s}{2}+1\right)+\frac{1}{\pi}\Delta_L\arg\zeta(s).$$

$$\Delta_L\arg(s-1)=\arg\left(-\tfrac{1}{2}+iT\right)=\frac{\pi}{2}+O\left(\tfrac{1}{T}\right)$$
$$\Delta_L\arg\pi^{-s/2}=\Delta_L\arg e^{-\frac{s}{2}\log\pi}=-\frac{T}{2}\log\pi$$

$$\log\Gamma(z)=\left(z-\tfrac{1}{2}\right)\log z-z+\log\sqrt{2\pi}+O\left(\tfrac{1}{|z|}\right).$$

$$\Delta_L\arg\Gamma\left(\tfrac{s}{2}+1\right)=\operatorname{Im}\log\Gamma\left(\tfrac{5}{4}+i\tfrac{T}{2}\right)=$$
$$=\operatorname{Im}\left[\left(\tfrac{3}{4}+i\tfrac{T}{2}\right)\log\left(\tfrac{5}{4}+i\tfrac{T}{2}\right)-\tfrac{5}{4}-i\tfrac{T}{2}+O\left(\tfrac{1}{T}\right)\right]=$$
$$=\frac{3}{4}\left(\frac{\pi}{2}+O\left(\tfrac{1}{T}\right)\right)+\frac{T}{2}\log\sqrt{\frac{T^2}{4}+\frac{25}{16}}-\frac{T}{2}+O\left(\tfrac{1}{T}\right)=$$
$$=\frac{3}{8}\pi+\frac{T}{2}\log\frac{T}{2}+\frac{T}{4}\log\left(1+\frac{25}{4T^2}\right)-\frac{T}{2}+O\left(\tfrac{1}{T}\right)=$$

$$\sqrt{\frac{T^2}{4}+\frac{25}{16}}=\frac{T}{2}\sqrt{1+\frac{25}{4T^2}}$$

$$=\frac{3}{8}\pi+\frac{T}{2}\log\frac{T}{2}-\frac{T}{2}+O\left(\tfrac{1}{T}\right).$$

$$\Delta_L\arg(s-1)+\Delta_L\arg\pi^{-s/2}+\Delta_L\arg\Gamma\left(\tfrac{s}{2}+1\right)=$$
$$=\frac{7}{8}\pi+\frac{T}{2}\log\left(\frac{T}{2\pi}\right)-\frac{T}{2}+O\left(\tfrac{1}{T}\right)\Longrightarrow$$

$\Rightarrow N(T)=\dfrac{T}{2\pi}\log\left(\dfrac{T}{2\pi}\right)+\dfrac{7}{8}-\dfrac{T}{2\pi}+S(T)+O\left(\tfrac{1}{T}\right)$ dove
$S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right)$.

Lemma: si ha $S(T)\ll\log T$.

Dim.: $\arg\zeta(2+iT)-\arg\zeta(2)=\arg\zeta(2+iT)$
$$|\zeta(2+iT)|=\left|\sum_{n=1}^{+\infty}\frac{1}{n^{2+iT}}\right|\geq 1-\left|\sum_{n=2}^{+\infty}\frac{1}{n^{2+iT}}\right|\geq 1-\sum_{n=2}^{+\infty}\frac{1}{|n^{2+iT}|}=$$
$$=1-\sum_{n=2}^{+\infty}\frac{1}{n^2}=1-\left(\frac{\pi^2}{6}-1\right)>\frac{1}{3}.$$
$$\operatorname{Re}\zeta(2+iT)=1+\sum_{n=2}^{+\infty}\operatorname{Re}\frac{1}{n^{2+iT}}\geq 1-\sum_{n=2}^{+\infty}\frac{1}{n^2}>\frac{1}{3}\Rightarrow$$
$$\Rightarrow |\arg\zeta(2+iT)|\leq\pi/2.$$

$\arg\zeta\left(\frac{1}{2}+iT\right)-\arg\zeta(2+iT)$?

$$m=\#\left\{\sigma_j\in\left[\tfrac{1}{2},2\right] \mid \operatorname{Re}\zeta(\sigma_j+iT)=0\right\}$$
$$\left|\arg\zeta\left(\tfrac{1}{2}+iT\right)-\arg\zeta(2+iT)\right|\leq(m+1)\pi$$

$f(s)=\zeta(s+iT)+\zeta(s-iT)$ motivi di coniugio
$$f(\sigma)=\zeta(\sigma+iT)+\zeta(\sigma-iT)=2\operatorname{Re}\zeta(\sigma+iT)$$
$$m=\#\left\{\sigma_j\in\left[\tfrac{1}{2},2\right] \mid f(\sigma_j)=0\right\}$$
$$m\leq M=\#\left\{s\in\mathbb{C} \mid |s-2|\leq\tfrac{3}{2},\ f(s)=0\right\}$$

Per un vecchio Cor. con $r=\frac{3}{2}$ e $R=\frac{7}{4}$,
$$M\leq\frac{1}{\log\left(\frac{R}{r}\right)}\log\left(\frac{\max_{|s-2|\leq\frac{7}{4}}|f(s)|}{f(2)}\right)\leq\frac{1}{\log\left(\frac{7}{6}\right)}\log\left(\frac{\max_{|s-2|\leq\frac{7}{4}}2|\zeta(s+iT)|}{2/3}\right)\leq$$

$|\zeta(\sigma+iT)|\ll T$

$$\leq C_1\log T+C_2\ll\log T. \qquad\square\quad\square$$