$$N(T)-\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)+\frac{T}{2\pi}=\frac{7}{8}+S(T)+O\left(\frac{1}{T}\right)$$

$$S_1(T)=\int_0^T S(t)\,dt \ll \log T$$ (cioè S cambia spesso di segno)

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\left[N(t)-\frac{t}{2\pi}\log\left(\frac{t}{2\pi}\right)-\frac{t}{2\pi}\right]dt=\frac{7}{8}.$$

Cor.: $N(T+H)-N(T)\overset{\text{uniformemente}}{\ll}(H+1)\log(T+H)$. Inoltre, $N(T+H)-N(T)\gg H\log T$ se $H\geq H_0$.

Dim.: $$N(T+H)-N(T)=\frac{T+H}{2\pi}\log\left(\frac{T+H}{2\pi}\right)-\frac{T+H}{2\pi}-\left(\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}\right)+O(\log(T+H))=$$
$$=\int_{\frac{T}{2\pi}}^{\frac{T+H}{2\pi}}\log t\,dt+O(\log(T+H))\underset{0\leq\delta\leq 1}{=}\frac{H}{2\pi}\log\left(\frac{T+\delta H}{2\pi}\right)+O(\log(T+H))=$$
$$=\begin{cases}\ll \dfrac{H+1}{2\pi}\log(T+H)\\[2ex] >\dfrac{H}{2\pi}\log\dfrac{T}{2\pi}+O(\log(T+H))\gg H\log T.\ \square\end{cases}$$

Cor.: se $\rho_n=\beta_n+i\gamma_n$ sono gli zeri non banali di ζ ordinati per parte immaginaria crescente ($\gamma_n>0$) contati con molteplicità, allora si ha $\gamma_n\sim\dfrac{2\pi n}{\log n}$.

(Littlewood: $\gamma_{n+1}-\gamma_n\ll\dfrac{1}{\log\log\log\gamma_n}$)

Dim.: $$N(\gamma_n)=n\Rightarrow N(\gamma_n+1)\geq n\geq N(\gamma_n-1)\sim\frac{\gamma_n-1}{2\pi}\log\left(\frac{\gamma_n-1}{2\pi}\right)\sim$$
$$\sim\frac{\gamma_n}{2\pi}\log\gamma_n \qquad N(\gamma_n+1)\sim\frac{\gamma_n+1}{2\pi}\log\left(\frac{\gamma_n+1}{2\pi}\right)\sim\frac{\gamma_n}{2\pi}\log\gamma_n\Rightarrow$$
$$\Rightarrow\quad n\sim\frac{\gamma_n}{2\pi}\log\gamma_n\Rightarrow\log n\sim\log\gamma_n+\log\log\gamma_n-\log(2\pi)\sim$$
$$\sim\log\gamma_n\Rightarrow$$
$$\Rightarrow\gamma_n\sim\frac{2\pi n}{\log\gamma_n}\sim\frac{2\pi n}{\log n}.\ \square$$

Cor.: $\rho_n=\beta_n+i\gamma_n$ ha esponente di convergenza 1.

Dim.: $$\sum_{n=1}^{+\infty}\frac{1}{|\rho_n|}\underset{|\beta_n|\leq 1}{>}\sum_{n=1}^{+\infty}\frac{1}{1+|\gamma_n|}\geq\sum_{n=1}^{+\infty}\frac{1}{1+\frac{cn}{\log n}}=\sum_{n=1}^{+\infty}\frac{\log n}{\log n+cn}=+\infty$$
$$\sum_{n=1}^{+\infty}\frac{1}{|\rho_n|^{1+\varepsilon}}\leq\sum_{n=1}^{+\infty}\frac{1}{|\gamma_n|^{1+\varepsilon}}\leq c_2\sum_{n=1}^{+\infty}\frac{(\log n)^{1+\varepsilon}}{n^{1+\varepsilon}}\leq c_3\sum_{n=1}^{+\infty}\frac{1}{n^{1+\varepsilon/2}}<+\infty.\ \square$$

Le funzioni intere $\xi(s)$ e $(s-1)\zeta(s)$.

$\xi(s)=\dfrac{s(s-1)}{2}\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Per $\sigma>1$ ok. Ci manca $\sigma\geq 1/2$.

Si ha $\zeta(s)\ll|s|$ per $\sigma\geq 1/2$, poi
$$\Gamma(s)\ll_\varepsilon e^{|s|^{1+\varepsilon}}\ \forall\,\varepsilon>0,\ \pi^{-s/2}\ll 1,\ \frac{s(s-1)}{2}\ll|s^2|\Rightarrow$$
$\Rightarrow\quad \xi(s)\ll_\varepsilon e^{|s|^{1+\varepsilon}}$ per $\sigma\geq 1/2$ e da $\xi(s)=\xi(1-s)$ anche $\forall\,\sigma$.

$$\xi(s)=e^{a+As}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}.\quad a=\log(\xi(0))=\log\frac{1}{2}\Rightarrow$$
$$\Rightarrow\xi(s)=\frac{1}{2}e^{As}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}.$$

$$(s-1)\zeta(s)=\xi(s)\frac{\pi^{s/2}}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)}\Rightarrow \text{ha ordine } 1.$$

$$(s-1)\zeta(s)=e^{b+Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{n=1}^{+\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}$$
$$b=\left(\log((s-1)\zeta(s))\right)_{s=0}=\log\frac{1}{2}\Rightarrow$$
$$\Rightarrow(s-1)\zeta(s)=\frac{1}{2}e^{Bs}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{n=1}^{+\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}.$$
$$(s-1)\zeta(s)=\xi(s)\frac{\pi^{s/2}}{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)}=\frac{1}{2}e^{As}\prod_\rho\left(1-\frac{1}{\rho}\right)e^{\frac{s}{\rho}}e^{\frac{s}{2}\log\pi+\frac{s}{2}\gamma}\prod_{n=1}^{+\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}\Rightarrow$$
$$\Rightarrow B=A+\frac{1}{2}\log\pi+\frac{\gamma}{2}.$$
$$A=\frac{\xi'}{\xi}(0)=2\xi'(0),\quad B=\left[\frac{1}{s-1}+\frac{\zeta'}{\zeta}(0)\right]_{s=0}=-2\zeta'(0)-1.$$
$$\frac{B}{2}=\frac{A}{2}+\frac{1}{4}\log\pi+\frac{\gamma}{4}\Rightarrow$$
$$\Rightarrow\quad\xi'(0)+\zeta'(0)=-\frac{1}{2}-\frac{1}{4}\log\pi-\frac{\gamma}{4}.$$

$$\xi(s)=\frac{s(s-1)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$
$$\xi'(s)=\frac{(s-1)\zeta(s)}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)-\frac{1}{4}\log\pi\, s(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2}+\frac{s(s-1)}{4}\pi^{-s/2}\Gamma'\left(\frac{s}{2}\right)\zeta(s)+$$
$$+\frac{s}{2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)D(\zeta(s)(s-1))$$

$\zeta(s)=\dfrac{1}{s-1}+\gamma+(\cdots)$

$$\xi'(1)=\frac{\Gamma(1/2)}{2\sqrt{\pi}}-\frac{1}{4}\log\pi\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}}+\frac{\Gamma'\left(\frac{1}{2}\right)}{4\sqrt{\pi}}+\frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{\pi}}\gamma$$

$$\zeta(s)=\frac{1}{s-1}+1-s\int_1^{+\infty}\frac{\{u\}\,du}{u^{s+1}}\Rightarrow$$
$$\Rightarrow\lim_{s\to1}\left[\zeta(s)-\frac{1}{s-1}\right]=\gamma$$

$$\frac{\Gamma'}{\Gamma}(z)=-\gamma-\frac{1}{z}+\sum_{n=1}^{+\infty}\frac{z}{n(z+n)}\Rightarrow$$
$$\Rightarrow\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right)=-\gamma-2+\sum_{n=1}^{+\infty}\frac{2}{2n(2n+1)}=-\gamma-2+2\sum_{n=1}^{+\infty}\left(\frac{1}{2n}-\frac{1}{2n+1}\right)=$$
$$=-\gamma-2+2\left(\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\dots\right)=-\gamma-2+2-2\log 2=-\gamma-2\log 2\Rightarrow$$
$\Rightarrow\Gamma'(1/2)=-\sqrt{\pi}(\gamma+2\log 2)$. Quindi
$$\xi'(1)=\frac{1}{2}-\frac{1}{4}\log\pi-\frac{\gamma}{4}-\frac{1}{2}\log 2+\frac{\gamma}{2}=\frac{\gamma}{4}+\frac{1}{2}-\frac{1}{4}\log(4\pi)$$

$$\xi(1-s)=\xi(s)\Rightarrow-\xi'(1-s)=\xi'(s)\Rightarrow$$
$$\Rightarrow\xi'(0)=-\xi'(1)=\frac{1}{4}\log(4\pi)-\frac{\gamma}{4}-\frac{1}{2}$$
$$\zeta'(0)=\frac{\gamma}{4}+\frac{1}{2}-\frac{1}{4}\log(4\pi)-\frac{1}{2}-\frac{\gamma}{4}-\frac{1}{4}\log\pi=-\frac{1}{2}\log\pi-\frac{1}{2}\log 2=-\frac{1}{2}\log(2\pi).$$

$$B=-2\zeta'(0)-1=\log(2\pi)-1.$$
$$A=2\xi'(0)=\frac{1}{2}\log(4\pi)-1-\frac{\gamma}{2}.$$

$$\frac{\xi'(s)}{\xi(s)}=A+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$
$$\frac{\zeta'}{\zeta}(s)=-\frac{1}{s-1}-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)+\frac{1}{2}\log\pi+\frac{\xi'}{\xi}(s)=$$
$$=-\frac{1}{s-1}+A+\frac{1}{2}\log\pi+\sum_\rho\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right).$$

Supponiamo ci sia uno zero in $1+it$.

Si ha $3+4\cos\theta+\cos(2\theta)\geq 0\ \forall\,\theta\in\mathbb{R}$. Infatti è
$$2\cos^2\theta+4\cos\theta+2=2(\cos\theta+1)^2.$$

$$\sigma>1,\ \log\zeta(s)=\log\left(\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\right)=-\sum_p\log\left(1-\frac{1}{p^s}\right)=$$
$$=\sum_p\sum_{m=1}^{+\infty}\frac{1}{mp^{ms}}$$
$$\log|\zeta(\sigma+it)|=\sum_p\sum_{m=1}^{+\infty}\frac{1}{mp^{\sigma m}}\cos(mt\log p)$$
$$3\log|\zeta(\sigma)|+4\log|\zeta(\sigma+it)|+\log|\zeta(\sigma+i2t)|=$$
$$=\sum_p\sum_{m=1}^{+\infty}\frac{1}{mp^{m\sigma}}\Big(3+4\cos(\underbrace{mt\log p}_{\theta})+\cos(\underbrace{2mt\log p}_{2\theta})\Big)\geq 0\Rightarrow$$

$\Rightarrow\zeta^3(\sigma)|\zeta(\sigma+it)|^4|\zeta(\sigma+i2t)|\geq 1$. Se ci fosse uno zero in $1+it$, il polo di $\zeta^3(\sigma)$ che verrebbe mangiato dallo zero alla quarta.