$\rho=\beta+i\gamma$ zero non banale $\Rightarrow 0<\beta<1$

$$a,b\in\mathbb{C},\quad \frac{\zeta(s)\zeta(s-a)\zeta(s-a-b)}{\zeta(2s-a-b)}=\sum_{n=1}^{+\infty}\frac{\sigma_a(n)\,\sigma_b(n)}{n^s}$$

$a=i\gamma,\ b=-i\gamma,\ \beta=1$ (per assurdo) $\Rightarrow$ $\ \boxed{\sigma_a(n)=\sum_{d\mid n}d^a}$

$$\Rightarrow \frac{\zeta^2(s)\zeta(s-i\gamma)\zeta(s+i\gamma)}{\zeta(2s)}=\sum_{n=1}^{+\infty}\frac{\sigma_{i\gamma}(n)\sigma_{-i\gamma}(n)}{n^s}=\sum_{n=1}^{+\infty}\frac{|\sigma_{i\gamma}(n)|^2}{n^s}$$

$|\sigma_{i\gamma}(n)|\geqslant 0 \Rightarrow$ in σ_0 $\dfrac{\zeta^2(s)\zeta(s-i\gamma)\zeta(s+i\gamma)}{\zeta(2s)}$ ha un polo

$s=1 \Rightarrow \zeta(1+i\gamma)\zeta(1-i\gamma)=0$ (per assurdo)

$s=-1 \Rightarrow \zeta(2s)=\zeta(-2)=0$ (è il primo polo che trovo)

Allora la serie deve esistere (e "funzionare") fino a $\sigma>-1$, ma per $s=1/2$

$$\frac{\zeta^2(1/2)\zeta(1/2+i\gamma)\zeta(1/2-i\gamma)}{\zeta(1)}=0,\quad \sigma_{i\gamma}(1)=1,\ |\sigma_{i\gamma}(n)|^2\geqslant 0$$

(Ingham, vedere il Titchmarsh)

$$\xi(s)=\frac{1}{2}e^{As}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},\quad (s-1)\zeta(s)=\frac{1}{2}e^{Bs}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}\prod_{n=1}^{+\infty}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}$$

$$\xi(s)=\frac{s(s-1)}{2}\Gamma(s/2)\pi^{-\frac{s}{2}}\zeta(s)=\Gamma\left(\frac{s}{2}+1\right)\pi^{-\frac{s}{2}}(s-1)\zeta(s)$$

$$\frac{\zeta'}{\zeta}(s)=-\frac{1}{s-1}+\underbrace{A+\frac{1}{2}\log\pi}_{>0}-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$

Prop. (de la Vallée-Poussin, 1899):
$\exists$ una costante $C_0>0$ t.c. se $\rho=\beta+i\gamma$, $\gamma>0$ è uno zero non banale, allora

$$\beta<1-\frac{C_0}{\log\gamma}.$$

Dim.:

$$-\operatorname{Re}\frac{\zeta'}{\zeta}(s)=\frac{\sigma-1}{(\sigma-1)^2+t^2}-A'+\frac{1}{2}\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)-\sum_{\rho}\operatorname{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right)$$

$t\geqslant 2,\ 1<\sigma\leqslant 2 \Rightarrow -\operatorname{Re}\dfrac{\zeta'}{\zeta}(\sigma+it)<$ uso la stima per $\dfrac{\Gamma'}{\Gamma}$

$$<C\log t-\sum_{\rho}\left(\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}+\frac{\beta}{\beta^2+\gamma^2}\right)<C\log t-\sum_{\rho}\frac{\sigma-\beta}{(\sigma-\beta)^2+(t-\gamma)^2}$$

$$t=\begin{cases}\gamma \Rightarrow -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+i\gamma)<C\log\gamma-\frac{1}{\sigma-\beta}\\ 2\gamma \Rightarrow -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+i2\gamma)<C\log(2\gamma)\leqslant C_1\log\gamma\end{cases}$$

$$t=0\Rightarrow -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma)=\frac{1}{\sigma-1}+O(1)$$

$$-\frac{\zeta'}{\zeta}(s)=\sum_{n=1}^{+\infty}\frac{\Lambda(n)}{n^s}$$

$$-\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it)=\sum_{n=1}^{+\infty}\frac{\Lambda(n)}{n^\sigma}\cos(t\log n)$$

$$3+4\cos(\gamma\log n)+\cos(2\gamma\log n)\geqslant 0\Rightarrow$$

$$\Rightarrow -3\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma)-4\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+i\gamma)-\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+i2\gamma)\geqslant 0\Rightarrow$$

$$\Rightarrow 0\leqslant\frac{3}{\sigma-1}-\frac{4}{\sigma-\beta}+C_2\log\gamma\Rightarrow$$

$$\Rightarrow \frac{4}{\sigma-\beta}\leqslant\frac{3}{\sigma-1}+C_2\log\gamma.$$

Prendiamo $\sigma=1+\dfrac{\delta}{\log\gamma}$, $\delta>0$.

$$\frac{4\cancel{\log\gamma}}{\delta+(1-\beta)\log\gamma}\leqslant\frac{3\cancel{\log\gamma}}{\delta}+C_2\cancel{\log\gamma}\Rightarrow$$

$$\Rightarrow\frac{4\delta}{\delta+(1-\beta)\log\gamma}\leqslant\frac{3+C_2\delta}{\cancel{\delta}}\Rightarrow \text{... fatti i conti...}\Rightarrow$$

$\Rightarrow$ tesi. $\square$

Oss.: $6\leqslant\gamma\leqslant t\Rightarrow 1-\dfrac{C_0}{\log\gamma}\leqslant 1-\dfrac{C_0}{\log t}$. La regione

$\sigma>1-\dfrac{C_0}{\log(|t|+2)}$ è libera da zeri di ζ.

Littlewood, 1922: $\sigma>1-\dfrac{C_0\log\log(|t|+2)}{\log(|t|+2)}$

Vinogradov, 1958: $\sigma>1-\dfrac{C_0(\varepsilon)}{(\log(|t|+2))^{2/3+\varepsilon}}\quad\forall\varepsilon>0$

$$-\operatorname{Re}\frac{\zeta'}{\zeta}(s)<C\log t-\sum_{\rho}\operatorname{Re}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right),\quad t\geqslant 2$$

$$s=2+it\Rightarrow \operatorname{Re}\sum_{\rho}\left(\frac{1}{(2-\beta)+i(t-\gamma)}+\frac{1}{\rho}\right)<C\log t$$

$$\sum_{\rho}\frac{1}{4+(t-\gamma)^2}\leqslant\sum_{\rho}\left(\frac{2-\beta}{(2-\beta)^2+(t-\gamma)^2}+\frac{\beta}{\beta^2+\gamma^2}\right)<C\log t$$

$$\sum_{|\gamma-t|\leqslant 1}\frac{1}{5}\leqslant\sum_{|\gamma-t|\leqslant 1}\frac{1}{4+(t-\gamma)^2}\leqslant C\log t\Rightarrow$$

$$\Rightarrow N(t+1)-N(t-1)\leqslant 5C\log t$$

$$\sum_{|\gamma-t|\geqslant 1}\frac{1}{4+(t-\gamma)^2}\leqslant C\log t$$

Lemma: se $-1\leqslant\sigma\leqslant 2$ si ha

$$\frac{\zeta'}{\zeta}(s)=\sum_{|\gamma-t|\leqslant 1}\frac{1}{s-\rho}+O(\log(|t|+2)).$$

Dim.: $\dfrac{\zeta'}{\zeta}(s)=\sum_{\rho}\left(\dfrac{1}{s-\rho}+\dfrac{1}{\rho}\right)+O(\log t)\qquad(|t|\geqslant 2)$

$$\frac{\zeta'}{\zeta}(2+it)=\sum_{\rho}\left(\frac{1}{2+it-\rho}+\frac{1}{\rho}\right)+O(\log t)$$

$$\frac{\zeta'}{\zeta}(s)=\underbrace{\sum_{\rho}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)}_{=\sum_{\rho}\frac{2-\sigma}{(s-\rho)(2+it-\rho)}}+O(\log t)$$

$$\left|\sum_{|\gamma-t|>1}\frac{2-\sigma}{(s-\rho)(2+it-\rho)}\right|\leqslant\sum_{|\gamma-t|>1}\frac{3}{(\gamma-t)^2}\ll\log t\Rightarrow$$

$$\Rightarrow\frac{\zeta'}{\zeta}(s)=\sum_{|\gamma-t|\leqslant 1}\left(\frac{1}{s-\rho}-\frac{1}{2+it-\rho}\right)+O(\log t)\quad\text{e}$$

$$\sum_{|\gamma-t|\leqslant 1}\frac{1}{|2+it-\rho|}\leqslant\sum_{|\gamma-t|\leqslant 1}\frac{1}{2-\beta}\leqslant N(t+1)-N(t-1)\ll\log t\Rightarrow\text{tesi.}$$

$\square$

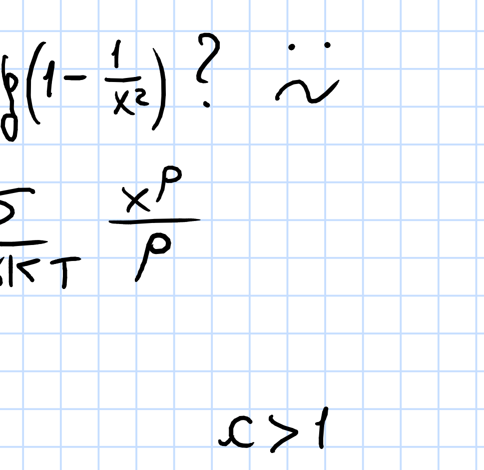
$$S(T)=\frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+iT\right)-\underbrace{\frac{1}{\pi}\arg\zeta(2+iT)}_{O(1)}=$$

$$=\frac{1}{\pi}\operatorname{Im}\left(\log\zeta\left(\tfrac{1}{2}+iT\right)-\log\zeta(2+iT)\right)=$$

$$=-\frac{1}{\pi}\int_{1/2}^{2}\operatorname{Im}\frac{\zeta'}{\zeta}(\sigma+iT)\,d\sigma=$$

$$=-\frac{1}{\pi}\int_{1/2}^{2}\sum_{|\gamma-T|\leqslant 1}\operatorname{Im}\left(\frac{1}{\sigma+iT-\rho}\right)d\sigma+O(\log T)=$$

$$=-\frac{1}{\pi}\sum_{|\gamma-T|\leqslant 1}\Delta_\ell\arg(s-\rho)+O(\log T)\ll\sum_{|\gamma-T|\leqslant 1}1+O(\log T)\ll\log T.$$



$$\psi(x)=\sum_{n\leqslant x}\Lambda(n)\overset{(\star)}{=}x-\sum_{\rho}\frac{x^\rho}{\rho}-\frac{\zeta'}{\zeta}(0)-\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)\ ?\ \sim$$

$\sum_{\rho}\frac{x^\rho}{\rho}$ $\longrightarrow$ $\lim_{T\to+\infty}\sum_{|\gamma|<T}\dfrac{x^\rho}{\rho}$

$$\psi_0(x)=\sum_{n<x}\Lambda(n)+\frac{1}{2}\Lambda(x)$$

$\hookrightarrow \Lambda=0$ fuori dagli interi

$$\psi_0(x)\sim\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}-\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}\,ds\qquad c>1$$

$\uparrow$ l'idea di Riemann

$\downarrow$ i poli ci danno i vari termini in $(\star)$