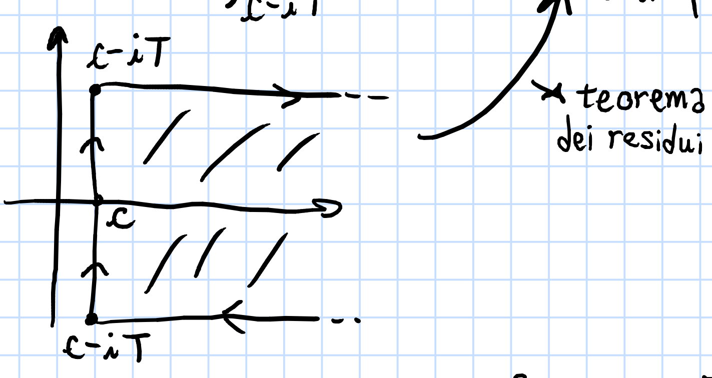
Lemma: $c>0$, sia $\frac{1}{2\pi i}\int_{c-iT}^{c+iT} y^s \frac{ds}{s} = I(y,T)$; allora

$$|I(y+T)-\delta(y)| \leq \begin{cases} y^c \min\left\{1, \frac{1}{T|\log y|}\right\} & \text{se } y\neq 1 \\ \frac{c}{T} & \text{se } y=1 \end{cases}$$

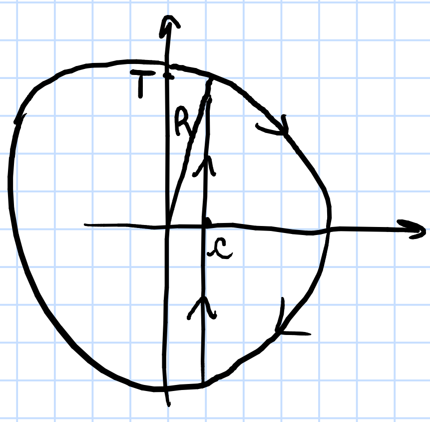
dove $\delta(y)=\begin{cases} 1 & y>1 \\ \frac{1}{2} & y=1 \\ 0 & 0<y<1 \end{cases}$.

Cor.: $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \delta(y)$.

Dim.: $0<y<1$, $\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\,ds = \frac{1}{2\pi i}\left(\int_{c-iT}^{+\infty-iT} - \int_{c+iT}^{+\infty+iT}\right) y^s \frac{ds}{s} = J_1 - J_2$



$$|J_1| \leq \frac{1}{T}\int_c^{+\infty} y^\sigma\,d\sigma = \frac{1}{T}\frac{y^c}{|\log y|}, \quad J_2 \text{ idem.}$$

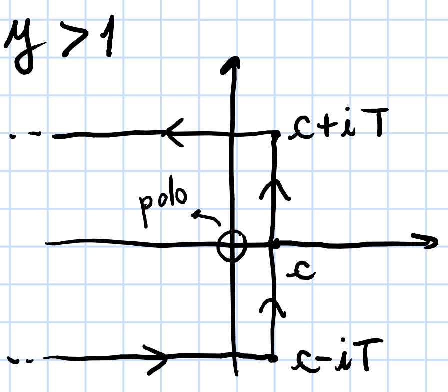


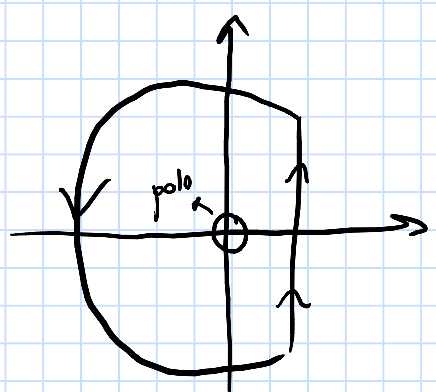
$R=\sqrt{c^2+T^2} \Rightarrow$

$$\Rightarrow \left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{y^s}{s}ds\right| = \left|\frac{1}{2\pi i}\int_\gamma \frac{y^s}{s}ds\right| \leq$$

$$\leq \frac{y^c}{2\pi}\int_\gamma \frac{d|s|}{|s|} \leq y^c\frac{\pi R}{2\pi R} \leq \frac{y^c}{2}.$$

$y>1$





$\operatorname{Res}_{s=0} \frac{y^s}{s} = 1$ e poi le stime sono analoghe.

$y=1$: $\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{ds}{s} = \frac{1}{2\pi \cancel{i}}\int_{-T}^{T}\frac{\cancel{i}\,dt}{c+it} = \frac{1}{2\pi}\int_{-T}^{T}\frac{c-it}{c^2+t^2}dt =$

$$= \frac{1}{2\pi}\int_{-T}^{T}\frac{c}{c^2+t^2}dt = \frac{1}{\pi}\int_0^T\frac{c}{c^2+t^2}dt \underset{\frac{t}{c}=x}{=} \frac{1}{\pi}\int_0^{T/c}\frac{dx}{x^2+1} =$$

$$= \frac{1}{\pi}\left(\int_0^{+\infty}\frac{dx}{x^2+1} - \int_{T/c}^{+\infty}\frac{dx}{x^2+1}\right) \leq \frac{1}{2}+\frac{c}{T}. \quad \square$$

$y=\frac{x}{m}$, $m\in\mathbb{N}$

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{\Lambda(m)}{m^s}x^s\frac{ds}{s} = \Lambda(m)\cdot\begin{cases} 1+O\left(\left(\frac{x}{m}\right)^c\min\left\{1,\frac{1}{T|\log\frac{x}{m}|}\right\}\right) & m<x \\ \frac{1}{2}+O\left(\frac{c}{T}\right) & m=x \ (\text{e } m=p^a\ldots) \\ O\left(\left(\frac{x}{m}\right)^c\min\left\{1,\frac{1}{T|\log\frac{x}{m}|}\right\}\right) & \text{se } m>x \end{cases}$$

$$\sum_{m\leq x}\Lambda(m)+\frac{\Lambda(x)}{2} \underset{c>1}{=} \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\left(\sum_{m=1}^{+\infty}\frac{\Lambda(m)}{m^s}\right)\frac{x^s}{s}ds + O\left(\sum_{\substack{m=1\\m\neq x}}^{+\infty}\frac{\Lambda(m)x^c}{m^c}\min\{1,\ldots\}+\frac{c\Lambda(x)}{T}\right)$$

Stima del resto: prendiamo $c=1+\frac{1}{\log x} \Rightarrow x^c = ex \ll x$.

- $\frac{c\Lambda(x)}{T} \ll \frac{\log x}{T}$
- $m\leq\frac{3}{4}x \vee m\geq\frac{5}{4}x \Rightarrow \left|\log\left(\frac{x}{m}\right)\right| \gg 1$

$$\left(\sum_{m\leq\frac{3}{4}x}+\sum_{m\geq\frac{5}{4}x}\right)\frac{\Lambda(m)}{m^c}\frac{x^c}{T} \ll \frac{x}{T}\sum_{m=1}^{+\infty}\frac{\Lambda(m)}{m^c} \underset{\substack{\zeta(\sigma)\ll\frac{1}{\sigma-1},\\ \frac{\zeta'}{\zeta}(\sigma)\ll\frac{1}{\sigma-1}}}{\ll} \frac{x\log x}{T}$$

- $\frac{3}{4}x<m<x$, sia $\frac{3}{4}x<x_1<x$ max tra le potenze di primo

$$\log\left(\frac{x}{x_1}\right) = -\log\left(\frac{x_1}{x}\right) = -\log\left(1-\frac{x-x_1}{x}\right) = \frac{x-x_1}{x}+\frac{1}{2}\left(\frac{x-x_1}{x}\right)^2+\ldots > \frac{x-x_1}{x}$$

$$\Lambda(x_1)\left(\frac{x}{x_1}\right)^c\frac{x}{T(x-x_1)} \ll \frac{x\log x}{T(x-x_1)}$$

E $\frac{3}{4}x<m<x_1$? $\log\left(\frac{x}{m}\right) \geq \log\left(\frac{x_1}{m}\right) = -\log\left(\frac{m}{x_1}\right) = -\log\left(1-\frac{x_1-m}{x_1}\right) \geq$

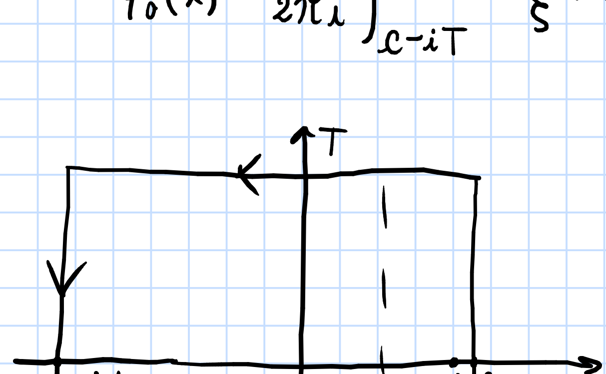
$$\geq \frac{x_1-m}{x_1} = \frac{\nu}{x_1}$$

$$\sum_{1\leq\nu<x_1}\frac{\Lambda(x_1-\nu)x^c}{\underbrace{(x_1-\nu)^c}_{\substack{\parallel \\ x>x_1>m\geq\frac{3}{4}x}}}\frac{x_1}{T\nu} \ll \frac{x(\log x)^2}{T}$$

- $x<m<\frac{5}{4}x$ é analogo, con x_2 la min potenza di primo.

Ora, definendo $\langle x\rangle$ = distanza di x dalla più vicina potenza di primo, si ha $\psi_0(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds + O\left(\frac{x\log^2 x}{T}+\frac{x\log x}{T\langle x\rangle}\right)$.

$$\psi_0(x) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds + \underbrace{R(x,T)}_{\substack{\Uparrow\\ \frac{x\log^2 x}{T}\ (x\in\mathbb{N})}}$$



$T\neq\gamma$ (non becchiamo gli zeri), residui $\Rightarrow$

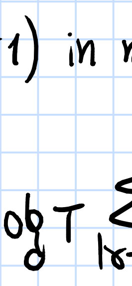
$$\Rightarrow \psi_0(x) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) +$$

$$-\sum_{1\leq m<\frac{U}{2}}\frac{x^{-2m}}{-2m} + \frac{1}{2\pi i}\left(\int_{-U+iT}^{c+iT} - \int_{-U-iT}^{c-iT} + \int_{-U-iT}^{-U+iT}\right)(\ldots) +$$

$$+R(x,T)$$

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-t|\leq 1}\frac{1}{s-\rho}+O(\log t) \qquad (|t|\geq 2, -1\leq\sigma\leq 2)$$

$$N(T+1)-N(T-1)\leq C_0\log T$$



$2C_0\log T$ intervalli $\Rightarrow$

$\Rightarrow$ ce n'è almeno uno senza zeri. Allora scegliamo T (variandolo di una quantità ≤ 1) in modo che

$$|\gamma-T| \gg \frac{1}{\log T}.$$

$$\frac{\zeta'}{\zeta}(\sigma+iT) \ll \sum_{|\gamma-T|\leq 1}\frac{1}{|s-\rho|} + O(\log T) \ll \log T\sum_{|\gamma-T|\leq 1} 1 \ll \log^2 T \Rightarrow$$

$$\Rightarrow \int_{-1\pm iT}^{c\pm iT} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds \ll \frac{\log^2 T\, x^c}{T} \ll \frac{x\log^2 T}{T}.$$

Oss.: per $\sigma\leq -1$ e $|s+2m|\geq\frac{1}{2}$ $\forall m\in\mathbb{N}$ si ha

$$\frac{\zeta'}{\zeta}(s) \ll \log(2|s|) \Rightarrow$$

$$\Rightarrow \log(2|U\pm iT|)x^{-U}\int_0^T\frac{dt}{U+t} \ll \frac{\log(2|U\pm iT|)}{x^U}\log T \xrightarrow[U\to+\infty]{} 0$$

$$\frac{1}{T}\int_{-\infty}^{-1}\log(2|\sigma+iT|)x^\sigma d\sigma \ll \frac{x^{-1}\log T}{T\log x} \quad \text{trascurabile.}$$

Prop.: $\psi_0(x) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) + R(x,T)$

dove $R(x,T) \ll \frac{x\log^2(xT)}{T}$ $\quad (x\in\mathbb{N}, T\geq 2)$.

Mancano degli zeri: $\ll \frac{x^\rho}{\rho}\log T \ll \frac{x^\beta\log T}{T} \ll \frac{x\log T}{T}$.