Per $|s+2m| \geq 1/2$, $\sigma \leq -1$ si ha $\quad \zeta(s) \ll \log(2|s|)$.

Dim.: $\xi(s) = \frac{s(s-1)}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$

$\qquad \|$

$\xi(1-s) = \frac{s(s-1)}{2}\pi^{\frac{s-1}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \quad \Longrightarrow$

$$\Longrightarrow \zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s)$$

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \qquad \forall z \in \mathbb{C} \qquad \Longrightarrow$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

$$\Longrightarrow \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)} = \frac{\sqrt{\pi}\,2^{1-s}\,\Gamma(s)}{\pi/\sin\left(\frac{\pi}{2}-\frac{\pi s}{2}\right)} = 2^{1-s}\pi^{-1/2}\cos\left(\frac{\pi s}{2}\right)\Gamma(s) \Longrightarrow$$

$$\Longrightarrow \zeta(1-s) = 2^{1-s}\pi^{-s}\cos\left(\frac{\pi}{2}s\right)\Gamma(s)\zeta(s) \Longrightarrow$$

$$\Longrightarrow \frac{\zeta'}{\zeta}(1-s) = \log 2 + \log\pi + \frac{\pi}{2}\tan\left(\frac{\pi}{2}s\right) - \frac{\Gamma'}{\Gamma}(s) - \frac{\zeta'}{\zeta}(s).$$

Per $\sigma \geq 2$, cioè $1-\sigma \leq -1$, si ha

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{+\infty}\frac{\Lambda(n)}{n^s} \ll 1.$$

$$\left|\tan\left(\frac{\pi}{2}s\right)\right| = \left|\frac{e^{i\frac{\pi}{2}s} - e^{-i\frac{\pi}{2}s}}{i\left(e^{i\frac{\pi}{2}s}+e^{-i\frac{\pi}{2}s}\right)}\right| = \left|\frac{e^{\pi i s}-1}{e^{\pi i s}+1}\right| \leq \frac{e^{\pi t}+1}{e^{\pi t}-1} =$$

$$= 1 + \underbrace{\frac{2}{e^{\pi t}-1} \leq 3}_{t \geq 1/2 \text{ fuori dai } -2n}$$

$$\frac{\Gamma'}{\Gamma}(s) \ll \log|s| + O\left(\frac{1}{s}\right). \qquad |1-s| \leq 1+|s| \leq 2|s|.$$

Allora $\quad \frac{\zeta'}{\zeta}(1-s) \ll \log(2|1-s|)$.

$$\psi_0(x) = \sum_{n<x}\Lambda(n) + \frac{1}{2}\Lambda(x) \Longrightarrow$$

$$\Longrightarrow \psi_0(x) = x - \sum_{|\gamma|<T}\frac{x^\rho}{\rho} + O\left(\frac{x\log^2(xT)}{T} + \log x \cdot \min\left\{1, \frac{x}{T\langle x\rangle}\right\}\right).$$

Per de la Vallée-Poussin si ha che $\exists\ c_0 > 0$ t.c.

$$\beta < 1 - \frac{c_0}{\log T} \quad \forall \rho = \beta + i\gamma \text{ con } |\gamma| < T \Longrightarrow$$

$$\Longrightarrow \left|\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right| \leq \max_{|\gamma|<T} x^\beta \sum_{|\gamma|<T}\frac{1}{|\rho|} \leq x\exp\left(-\frac{c_0\log x}{\log T}\right)\sum_{|\gamma|<T}\frac{1}{|\rho|}$$

$$\sum_{|\gamma|<T}\frac{1}{|\rho|} \underset{\rho=\sqrt{\beta^2+\gamma^2}\geq|\gamma|}{\ll} \sum_{2\leq\gamma\leq T}\frac{1}{\gamma} \underset{\text{sommazione parziale}}{=} \left(\sum_{2\leq\gamma\leq T}1\right)\frac{1}{T} - \int_2^T\left(\sum_{2\leq\gamma\leq u}1\right)\frac{du}{u^2} =$$

$$= \frac{N(T)}{T} + \int_2^T\frac{N(u)}{u^2}\,du \underset{\text{Riemann--Von Mangoldt}}{\ll} \frac{T\log T}{T} + \int_2^T\frac{\log u}{u}\,du \ll$$

$$\ll \log T + \log^2 T \ll \log^2 T. \quad \text{Otteniamo}$$

$$\sum_{|\gamma|<T}\frac{x^\rho}{\rho} \ll x\log^2 T\exp\left(-\frac{c_0\log x}{\log T}\right) \Longrightarrow$$

$$\Longrightarrow \psi_0(x) - x \ll x\left(\log^2 T\exp\left(-\frac{c_0\log x}{\log T}\right) + \frac{\log^2(xT)}{T}\right).$$

Prendendo $T = \exp(\sqrt{\log x})$, $\log x \asymp \log^2 T \Longrightarrow$

$\Longrightarrow \exists\ 0 < c_1 < 1$ t.c.

$$\psi_0(x) - x \ll x\exp\left(-c_1\sqrt{\log x}\right).$$

Prop. (PNT) $\exists\ c_1$ t.c. $0 < c_1 < 1$ e

$$\psi(x) = x + O\left(x\exp\left(-c_1\sqrt{\log x}\right)\right).$$

Oss.: Littlewood, 1922: $\psi(x) = x + O\left(x\exp\left(-c_1\sqrt{\log x\log\log x}\right)\right) \quad (1)$

Vinogradov-Korobov, 1958: $\psi(x) = x + O_\varepsilon\left(x\exp\left(-c_1(\log x)^{3/5-\varepsilon}\right)\right) \quad (2)$

$\beta < 1 - \frac{c_0\log\log T}{\log T} \Longrightarrow (1)$

$\beta < 1 - \frac{c_0}{(\log T)^{2/3+\varepsilon}} \Longrightarrow (2)$

(si fanno come sopra)

RH $\Longrightarrow \psi(x) = x + O(x^{1/2}\log^2 x)$. Infatti

$$\left|\sum_{|\gamma|<T}\frac{x^\rho}{\rho}\right| \ll \sqrt{x}\sum_{|\gamma|<T}\frac{1}{|\gamma|} \ll \sqrt{x}\log^2 T \text{ e si prende } T = \sqrt{x},$$

poi come sopra.

Quasi RH: $\theta = \sup_\rho \beta$, $\frac{1}{2} < \theta < 1 \overset{\text{Ex.}}{\Longrightarrow} \psi(x) = x + O(x^\theta\log^2 x)$.

Montgomery (congettura): $x^{1/2}(\log\log x)^2$. $x^{1/2}$ no:

$\psi(x) - x = \Omega_\pm(\sqrt{x})$, cioè

$\psi(x) - x \geq c\sqrt{x}$ e $\psi(x) - x \leq -c\sqrt{x}$ freq..

Oss.: se $\psi(x) = x + O_\varepsilon(x^{\theta+\varepsilon})$ $\forall\, \varepsilon > 0$, allora $\beta \leq \theta$ $\forall\, \rho$.

$$\sigma > 1, \quad \sum_{n\leq N}\frac{\Lambda(n)}{n^s} \underset{\text{sommazione parziale}}{=} \frac{\psi(N)}{N^s} + s\int_2^N\frac{\psi(u)}{u^{s+1}}\,du \Longrightarrow$$

$$\Longrightarrow -\frac{\zeta'}{\zeta}(s) = s\int_1^{+\infty}\frac{\psi(u)}{u^{s+1}}\,du = s\int_1^{+\infty}u^{-s}\,du + s\int_1^{+\infty}\frac{R(u)}{u^{s+1}}\,du =$$

$R(u) = O_\varepsilon(u^{\theta+\varepsilon})$

$$= \frac{s}{s-1} + \int_1^{+\infty}(\dots)\,du \underset{\sigma>\theta+2\varepsilon}{\ll_\varepsilon} |s| + \frac{1}{|s-1|} \Longrightarrow \text{no poli (tranne 1)} \Longrightarrow$$

$\Longrightarrow$ no zeri per $\zeta \Longrightarrow \beta \leq \theta$.

$$\vartheta(x) = \sum_{p\leq x}\log p \Longrightarrow \psi(x) = \vartheta(x) + \vartheta(x^{1/2}) + \dots + \vartheta(x^{1/N}) \text{ con}$$

$x^{1/N} \geq 2 \Longrightarrow N \ll \log x$.

$\vartheta(y) \ll y \Longrightarrow \quad \psi(x) = \vartheta(x) + O(\sqrt{x}) + O(\sqrt[3]{x}\log x) = \vartheta(x) + O(\sqrt{x})$

RH $\Longrightarrow \vartheta(x) = x + O(\sqrt{x}\log^2 x)$

QRH $\Longrightarrow \vartheta(x) = x + O(x^\theta\log^2 x) \quad (1)$

RH $\Longrightarrow \pi(x) = \underbrace{\operatorname{li} x}_{\int_2^x\frac{du}{\log u}} + O(\sqrt{x}\log x)$ e analogamente per QRH. (2)

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x\frac{\vartheta(u)}{u\log^2 u}\,du \quad \text{e con (1) si ottiene (2)}$$

$f: \mathbb{R}^+ \to \mathbb{R}$ t.c. $\int_0^{+\infty}|f(x)|x^\sigma\frac{dx}{x} \overset{(*)}{<} +\infty$ per $\sigma \in \mathbb{R}$ fissato.

Si dice Trasformata di Mellin di f la seguente:

$$\hat{f}(s) = \int_0^{+\infty}f(x)x^s\frac{dx}{x} \quad \text{con } s = \sigma + it.$$

Es.: se $f(x) = e^{-x}$ e $\sigma > 0$, $\hat{f}(s) = \Gamma(s)$.

Oss.: $x = e^{-2\pi u}$, $\frac{dx}{x} = -2\pi\,du$.

$$\hat{f}(\sigma+it) = 2\pi\int_{-\infty}^{+\infty}\underbrace{f(e^{-2\pi u})e^{-2\pi u\sigma}}_{\varphi_\sigma(u)}e^{-2\pi i t u}\,du =$$

$$= 2\pi\int_{-\infty}^{+\infty}\varphi_\sigma(u)e^{-2\pi i t u}\,du = 2\pi\hat{\varphi}_\sigma(t).$$

$$\hat{f}(\sigma+it) = 2\pi\hat{\varphi}_\sigma(t).$$

Prop.: se f è di classe C^1 e soddisfa $(*)$, si ha

$$f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\hat{f}(s)x^{-s}\,ds.$$