Dim. (della prop. della volta scorsa):

$$\varphi_\sigma(u) = f(e^{-2\pi u})e^{-2\pi\sigma u} \Rightarrow \mathfrak{F}(\sigma+it) = 2\pi\widehat{\varphi_\sigma}(t)$$

$$\widehat{\widehat{\varphi_\sigma}}(u) = \varphi_\sigma(-u)$$

$$f(e^{-2\pi u})e^{-2\pi\sigma u} = \varphi_\sigma(u) = \widehat{\widehat{\varphi_\sigma}}(-u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathfrak{F}(\sigma+it)e^{2\pi i u t}\,dt \Rightarrow$$

$$\Rightarrow f(e^{-2\pi u}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathfrak{F}(\sigma+it)e^{2\pi(\sigma+it)u}\,dt = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\mathfrak{F}(s)e^{2\pi s u}\,ds,$$

$e^{-2\pi u} = x$. □

Oss.: se f è C^1 a tratti con limite destro e sinistro finiti nei punti di discontinuità, la prop. continua a valere purché si applichi a $\tilde{f}(x) = \begin{cases} f(x) & \text{se } x \text{ è regolare} \\ \dfrac{\lim_{y\to x^+} f(y) + \lim_{y\to x^-} f(y)}{2} & \text{se } f \text{ è discontinua in } x \end{cases}$ e interpretando l'integrale come limite.

Es.: 1) $f(x) = \begin{cases} 1 & \text{se } 0<x<1 \\ 1/2 & \text{se } x=1 \\ 0 & \text{se } x>1 \end{cases} \Rightarrow \mathfrak{F}(s) \overset{\sigma>0}{=} \dfrac{1}{2\pi i}\displaystyle\int_0^{+\infty} f(x)x^s\frac{dx}{x} =$

$$= \frac{1}{2\pi i}\int_0^1 x^s\frac{dx}{x} = \left[\frac{x^s}{s}\right]_0^1 = \frac{1}{s} \Rightarrow$$

$$\Rightarrow f(x) = \frac{1}{2\pi i}\lim_{T\to+\infty}\int_{\sigma-iT}^{\sigma+iT}\frac{x^{-s}}{s}\,ds. \quad x = 1/y \Rightarrow$$

$$\Rightarrow \frac{1}{2\pi i}\lim_{T\to+\infty}\int_{\sigma-iT}^{\sigma+iT}\frac{y^s}{s}\,ds = \begin{cases} 1 & \text{se } y>1 \\ 1/2 & \text{se } y=1 \\ 0 & \text{se } 0<y<1 \end{cases}.$$

2) $k \geqslant 1$, $f(x) = \begin{cases} \dfrac{(1-x)^k}{k!} & \text{se } 0<x\leqslant 1 \\ 0 & \text{se } x\geqslant 1 \end{cases}$. $\sigma>0$,

$$\mathfrak{F}(s) = \frac{1}{k!}\int_0^1 (1-x)^k x^s\frac{dx}{x} \underset{\text{per parti}}{=} \frac{1}{s\,k!}\left[(1-x)^k x^s\right]_0^1 + \frac{1}{s(k-1)!}\int_0^1 (x-1)^{k-1}x^s\,dx =$$

$$= \text{itero per parti} = \frac{1}{s(s+1)\dots(s+k)} \Rightarrow$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^{-s}}{s(s+1)\dots(s+k)}\,ds = f(x), \quad y = 1/x \Rightarrow$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{y^s}{s(s+1)\dots(s+k)}\,ds = \begin{cases} \frac{1}{k!}\left(1-\frac{1}{y}\right)^k & \text{se } y\geqslant 1 \\ 0 & \text{se } 0<y\leqslant 1 \end{cases}$$

3) $k\geqslant 1$, $f(x) = \begin{cases} (-\log x)^k/k! & 0<x\leqslant 1 \\ 0 & x\geqslant 1 \end{cases}$, $\sigma>0 \Rightarrow$

$$\mathfrak{F}(s) = \frac{1}{k!}\int_0^1 (-\log x)^k x^s\frac{dx}{x} = \text{itero per parti} = \frac{1}{s^{k+1}} \Rightarrow$$

$$\Rightarrow f(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{x^{-s}}{s^{k+1}}\,ds, \quad y = 1/x \Rightarrow$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{y^s}{s^{k+1}}\,ds = \begin{cases} (\log y)^k/k! & y\geqslant 1 \\ 0 & 0<y\leqslant 1 \end{cases}$$

$$k=1,\ y = x/n \overset{\sigma>1}{\Rightarrow} \sum_{n\leqslant x}\Lambda(n)\log\left(\frac{x}{n}\right) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left(\sum_{n=1}^{+\infty}\frac{\Lambda(n)}{n^s}\right)\frac{x^s}{s^2}\,ds =$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} -\frac{\zeta'}{\zeta}(s)\frac{x^s}{s^2}\,ds$$

Oss.: $\displaystyle\sum_{n\leqslant x}\Lambda(n)\log\left(\frac{x}{n}\right) \underset{\substack{\text{sommazione}\\ \text{di Abel}}}{=} \sum_{n\leqslant x}\Lambda(n)\cdot 0 + \int_2^x \sum_{n\leqslant u}\Lambda(u)\frac{du}{u} = \int_2^x \frac{\psi(u)}{u}\,du =: \psi_1(u).$

Oss.: posto $\psi_\ell(x) = \displaystyle\int_2^x \frac{\psi_{\ell-1}(u)}{u}\,du$, $\ell\geqslant 2$, si ha

$$\psi_k(x) = \frac{1}{k!}\sum_{n\leqslant x}\Lambda(n)\left(\log\left(\frac{x}{n}\right)\right)^k.$$

Prop.: si ha la seguente formula esplicita per ψ_1:

$$\psi_1(x) = x - \sum_\rho \frac{x^\rho}{\rho^2} - \frac{\zeta'}{\zeta}(0)\log x - \left(\frac{\zeta'}{\zeta}\right)'(0) - \frac{1}{4}\sum_{m=1}^{+\infty}\frac{x^{-2m}}{m^2}.$$

Dim.: teorema dei residui. □

Cor.: $\psi_1(x) = x + O(x\exp(-c\sqrt{\log x}))$. Dim.: de la V-P. □

$RH \Rightarrow x + O(\sqrt{x})$, $QRH \Rightarrow x + O(x^\theta)$.

$\psi_1(x) = x + o(x)$. Vediamolo: dalla formula esplicita per ψ_1 si ottiene

$$\left|\sum_\rho \frac{x^\rho}{\rho^2}\right| \leqslant \sum_\rho \frac{x^\beta}{|\rho|^2} \Rightarrow \frac{\psi_1(x)-x}{x} \ll \sum_\rho \frac{x^{\beta-1}}{|\rho|^2}.$$

$$\lim_{x\to+\infty}\sum_\rho \frac{x^{\beta-1}}{|\rho|^2} = \sum_\rho \frac{1}{|\rho|^2}\lim_{x\to+\infty}x^{\beta-1} \underset{\beta<1}{=} 0.$$

Oss.: $\displaystyle\int_x^{x+h}\frac{\psi(u)}{u}\,du \geqslant \frac{h}{x+h}\psi(x)$, $\displaystyle\int_{x-h}^x \frac{\psi(u)}{u}\,du \leqslant \frac{h}{x-h}\psi(x) \Rightarrow$

$$\Rightarrow (x-h)\frac{\psi_1(x)-\psi_1(x-h)}{h} \leqslant \psi(x) \leqslant (x+h)\frac{\psi_1(x+h)-\psi_1(x)}{h}, \quad x\geqslant 2,\ h\leqslant x.$$

Prop.: si ha $\psi(x) = x + o(x)$.

Dim.: $\psi_1(y) \sim y \Rightarrow (x-h)\dfrac{\psi_1(x)-\psi_1(x-h)}{h} \sim (x-h)\dfrac{x-x+h}{h} \sim x-h$,

$$(x+h)\frac{\psi_1(x+h)-\psi_1(x)}{h} \sim x+h.$$

Basta prendere $h = o(x)$ e dall'oss. si ha

$$x \sim f(x) \leqslant \psi(x) \leqslant g(x) \sim x \Rightarrow \psi(x) \sim x. \quad \square$$