Oss.: data $g(s)$ olo. in $\sigma_1<\sigma<\sigma_2$ e continua in $\sigma_1\leq\sigma\leq\sigma_2$ e t.c. $g(\sigma+it)\xrightarrow[|t|\to+\infty]{}0\ \forall\ \sigma_1\leq\sigma\leq\sigma_2$, si può dimostrare che $f(x)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}g(s)x^{-s}\,ds$ non dipende da σ e si ha $g(s)=\hat{f}(s)$.

Teorema (Landau, vedi cap. 3 del Titchmarsh): se si hanno $\phi(t)$, $\theta(t)$ t.c. $\phi(t)\nearrow+\infty$, $\theta(t)\searrow$, $0<\theta(t)\leq 1$, $\frac{\phi(t)}{\theta(t)}=o(e^{\phi(t)})$ e per $1-\theta(t)\leq\sigma\leq 2$ si ha $\zeta(s)\ll e^{\phi(t)}$, allora

$$\beta<1-\frac{\theta(2t+1)}{c_0\phi(2t+1)}.$$

In particolare, $\phi(t)=\log t$, $\theta(t)=\frac{1}{2}\Rightarrow\beta<1-\frac{1}{2c_0\log(2t+1)}$.

Littlewood: $\theta(t)=\frac{(\log\log t)^2}{\log t}$, $\phi(t)=A\log\log(t)\Rightarrow\beta<1-\frac{c\log\log T}{\log T}$.

Anche (Vinogradov): $\theta(t)=\frac{A}{(\log t)^{2/3-2\varepsilon}}$, $\phi(t)=(\log t)^{\varepsilon}$,

$$\zeta(s)\ll\exp((\log t)^{\varepsilon})\Rightarrow\beta<1-\frac{c}{(\log t)^{2/3-\varepsilon}}.$$

1) $\sum_{N<n\leq 2N}e^{2\pi i\alpha n}\ll\frac{1}{\|\alpha\|}$, $0<\alpha<1$, $\|\alpha\|=\min\{|\alpha-n|,\ n\in\mathbb{N}\cup\{0\}\}$

2) $\sum_{N<n\leq 2N}\Lambda(n)e^{2\pi i\alpha n}\ll\left(\frac{N}{\sqrt{q}}+\sqrt{Nq}+N^{3/4}\right)\log^4 N$ dove $\left|\alpha-\frac{a}{q}\right|\leq\frac{1}{q^2}$

$\hookrightarrow 2N+1=p_1+p_2+p_3$ (1930, Vinogradov)

3) $\sum_{N<n\leq 2N}n^{-it}$ Vinogradov, 1958 $\rightsquigarrow\zeta(\sigma+it)$ quando $1-\sigma\ll\frac{1}{(\log t)^{2/3-\varepsilon}}$

4) $\sum_{N<n\leq 2N}\frac{\Lambda(n)}{n^{it}}$? Touran: $\sum_{N<n\leq 2N}\frac{\Lambda(n)}{n^{it}}\ll\frac{N}{t^b}$ per $N\geq t^a\Rightarrow$

$\Rightarrow\beta<1-\frac{b^?}{a^?}$

Caratteri di Dirichlet

q intero >1: i caratteri modulo q sono i caratteri del gruppo abeliano $(\mathbb{Z}/q\mathbb{Z})^*$.

$$\chi:(\mathbb{Z}/q\mathbb{Z})^*\to\mathbb{C}\ \text{(in realtà, } |z|=1\text{)},\qquad\phi(q)\text{ caratteri.}$$

χ_0 carattere principale.

$$(\chi_1\cdot\chi_2)(n)=\chi_1(n)\cdot\chi_2(n).$$

$$\chi^{-1}(n)=\overline{\chi}(n).$$

$\chi:\mathbb{Z}\to\mathbb{C}$:

1) $\chi(nm)=\chi(n)\chi(m)\ \forall\ n,m\in\mathbb{Z}$
2) χ ha periodo q
3) $|\chi(n)|=1$ se $(n,q)=1$, 0 altrimenti.

Caso p^a, $p>2$ primo. $(\mathbb{Z}/p^a\mathbb{Z})^*$ è ciclico, g generatore.

ω t.c. $\omega^{\phi(p^a)}=1$. n intero, $\nu(n)$ t.c. $g^{\nu(n)}=[n]_{\mathbb{Z}/p^a\mathbb{Z}}$

$$\chi_{p^a}(n)=\omega^{\nu(n)}=e^{\frac{2\pi i m\nu(n)}{\phi(p^a)}}.$$

Il caso $(\mathbb{Z}/2\mathbb{Z})^*$ è banale.

Caso 2^2: $(\mathbb{Z}/4\mathbb{Z})^*$, $\omega=1$ χ_0

$\omega=-1=e^{\frac{2\pi i}{2}}$

$$\chi_4(n)=\begin{cases}1 & n\equiv 1\ (4)\\-1 & n\equiv 3\ (4)\\0 & n\text{ pari}\end{cases}$$

Caso 2^a, $a\geq 3$: $(\mathbb{Z}/2^a\mathbb{Z})^*\cong\mathbb{Z}/2\mathbb{Z}\times\underbrace{\mathbb{Z}/2^{a-2}\mathbb{Z}}_{g\text{ generatore, } g^{\nu_0'}\equiv[n]}$.

$$n\equiv(-1)^{\nu_0}5^{\nu_0'}\ (2^a),\quad\chi(n)=e^{\pi i m_0\nu_0+\frac{2\pi i m_0'\nu_0'}{2^{a-2}}}.$$

$m\longleftrightarrow$ carattere, $\nu\longleftrightarrow n$.

Caso $q=2^a p_1^{a_1}\cdots p_r^{a_r}$,

$$\chi_q(n)=e^{\pi i m_0\nu_0+\frac{2\pi i m_0'\nu_0'}{2^{a-2}}+\frac{2\pi i m_1\nu_1}{\phi(p_1^{a_1})}+\dots+\frac{2\pi i m_r\nu_r}{\phi(p_r^{a_r})}}.$$

4) $\sum_{n=1}^{q}\chi(n)=\begin{cases}0 & \text{se }\chi\neq\chi_0\\\phi(q) & \text{se }\chi=\chi_0\end{cases}$

5) $\sum_{\chi\bmod q}\chi(n)=\begin{cases}0 & \text{se }n\not\equiv 1\ (q)\\\phi(q) & \text{se }n\equiv 1\ (q)\end{cases}$

$\overline{\chi}(a)=\chi^{-1}(a)=\chi(a^{-1})$. 5) $\Rightarrow$ se $(a,q)=1$,

$$\frac{1}{\phi(q)}\sum_{\chi\bmod q}\overline{\chi}(a)\chi(n)=\begin{cases}1 & \text{se }n\equiv a\ (q)\\0 & \text{altrimenti}\end{cases}$$

Def.: sia χ carattere mod q, $\chi\neq\chi_0$. Si dice funzione L relativa al carattere χ la somma della seguente serie:

$$L(s,\chi)=\sum_{n=1}^{+\infty}\frac{\chi(n)}{n^s}\qquad(\sigma>1).$$

Prop.: se $\chi\neq\chi_0$ la serie converge uniformemente per $\sigma\geq\varepsilon>0$.

Dim.: $\sum_{n\leq N}\frac{\chi(n)}{n^s}\underset{\substack{\downarrow\\\text{sommazione parziale}}}{=}\left(\sum_{n\leq N}\chi(n)\right)N^{-s}+s\int_1^N\left(\sum_{n\leq u}\chi(u)\right)\frac{du}{u^{s+1}}$.

$\left|\sum_{n\leq N}\chi(n)\right|\leq q$, $\sigma>1\Rightarrow$

$\Rightarrow L(s,\chi)=s\int_1^{+\infty}\left(\sum_{n\leq u}\chi(u)\right)\frac{du}{u^{s+1}}$, e questo converge unif. in $\sigma\geq\varepsilon>0$. $\square$

Oss.: $\chi=\chi_0$, $\sum_{n=1}^{N}\frac{\chi_0(n)}{n^s}$; si ha $\sum_{n\leq N}\chi_0(n)\leq\left(\left\lfloor\frac{N}{q}\right\rfloor+1\right)\phi(q)\Rightarrow$

$\Rightarrow$ non si ottiene conv. unif. per $\sigma\geq\varepsilon>0$.

Oss.: per l'identità di Eulero si ha

$$L(s,\chi)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}\text{ per }\sigma>1\ \forall\ \chi\bmod q.$$

$$L(s,\chi_0)=\prod_p\left(1-\frac{\chi_0(p)}{p^s}\right)^{-1}=\prod_{p\nmid q}\left(1-\frac{1}{p^s}\right)^{-1}=$$

$$=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\prod_{p|q}\left(1-\frac{1}{p^s}\right)=\prod_{p|q}\left(1-\frac{1}{p^s}\right)\zeta(s)\Rightarrow$$

$\Rightarrow L(s,\chi_0)$ é mero. con polo semplice in $s=1$ con residuo $\prod_{p|q}\left(1-\frac{1}{p}\right)=\frac{\phi(q)}{q}$.

Altri zeri: $p^{-s}=1$, cioè $\sigma=0$ e $t=\frac{2k\pi}{\log p}$.

Oss.: $\log L(s,\chi)=-\sum_p\log\left(1-\frac{\chi(p)}{p^s}\right)=\sum_p\sum_{m=1}^{+\infty}\frac{\chi(p^m)}{m\,p^{ms}}$, $\sigma>1$

$$\frac{L'}{L}(s,\chi)=-\sum_{n=1}^{+\infty}\frac{\chi(n)\Lambda(n)}{n^s}$$

$$\frac{L'}{L}(s,\chi_0)=-\sum_{n=1}^{+\infty}\frac{\chi_0(n)\Lambda(n)}{n^s}=-\sum_{\substack{n=1,\\(n,q)=1}}^{+\infty}\frac{\Lambda(n)}{n^s}=-\sum_{n=1}^{+\infty}\frac{\Lambda(n)}{n^s}+\sum_{p^a\|n}\frac{\log p}{p^s}$$

$p^a\|q$?

$$\log L(s,\chi)=\sum_p\frac{\chi(p)}{p^s}+O\left(\sum_p\sum_{m\geq 2}\frac{1}{p^{\sigma m}}\right)=\sum_p\frac{\chi(p)}{p^s}+O\left(\sum_p\frac{1}{p^\sigma(p^\sigma-1)}\right)=$$

questo anche per $\sigma>0$; $\sigma>\frac{1}{2}$

$$=\sum_p\frac{\chi(p)}{p^s}+O(1).$$

$\sum_{\chi\bmod q}\log L(s,\chi)=\sum_{\chi\bmod q}\sum_p\frac{\chi(p)}{p^s}+O(\phi(q))$. Prendendo s reale,

$$\log\prod_{\chi\bmod q}|L(s,\chi)|=\underbrace{\sum_p}_{\downarrow\ +\infty}\left(\frac{1}{p^s}+\sum_{\chi\neq\chi_0}\frac{\chi(p)}{p^s}\right)+O(\phi(q))$$