Derivando la formula dell'ultima volta (*) in α,

$$2\pi i \sum_{n\in\mathbb{Z}} n e^{-\pi n^2 x + 2\pi i \alpha n} = -\frac{2\pi}{x^{3/2}} \sum_{n\in\mathbb{Z}} (n+\alpha) e^{-\pi(n+\alpha)^2/x}$$

$$\sum_{n\in\mathbb{Z}} n e^{-\pi n^2 x + 2\pi i \alpha n} = \frac{i}{x\sqrt{x}} \sum_{n\in\mathbb{Z}} (n+\alpha) e^{-\pi(n+\alpha)^2/x} \qquad (**)$$

Prop.: sia χ primitivo mod q e sia

$$\xi(s,\chi) = \left(\frac{\pi}{q}\right)^{-\frac{s+a}{2}} \Gamma\left(\frac{s+a}{2}\right) L(s,\chi) \quad \text{con}$$

$a = \begin{cases} 0 & \text{se } \chi(-1)=1 \\ 1 & \text{se } \chi(-1)=-1 \end{cases}$. Allora $\xi(1-s,\overline{\chi}) = \dfrac{i^a\sqrt{q}}{\tau(\chi)}\xi(s,\chi)$.

In particolare $\xi(s,\chi)$ è intera (di ordine 1).

Dim.: caso $\chi(-1)=\chi(1)$

$$\Gamma(s/2) = \int_0^{+\infty} e^{-u} u^{s/2}\frac{du}{u} \overset{u=\pi n^2 x/q}{=} \int_0^{+\infty} e^{-\pi n^2 x/q} x^{s/2} \frac{dx}{x}\left(\frac{q}{\pi}\right)^{-s/2} n^s \Rightarrow$$

$$\overset{\sigma>1}{\Rightarrow} \left(\frac{\pi}{q}\right)^{-s/2}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{+\infty}\frac{\chi(n)}{n^s} = \int_0^{+\infty} \underbrace{\sum_{n=1}^{+\infty} \chi(n) e^{-\pi n^2 x/q}}_{\frac{1}{2}\vartheta(x,\chi)} x^{s/2}\frac{dx}{x} =$$

$$= \frac{1}{2}\int_0^{+\infty} \vartheta(x,\chi) x^{s/2}\frac{dx}{x}, \text{ dove } \vartheta(x,\chi) = \sum_{n\in\mathbb{Z}} \chi(n) e^{-\pi n^2 x/q}$$

$\chi(-1)=\chi(1) \Rightarrow \chi(n)=\chi(-n)$, $\chi(0)=0$

$$\chi(n) = \frac{1}{\tau(\overline{\chi})}\sum_{m=1}^{q} \overline{\chi}(m) e^{2\pi i m n/q} \Rightarrow$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} \chi(n) e^{-\pi n^2 x/q} = \frac{1}{\tau(\overline{\chi})}\sum_{m=1}^{q}\overline{\chi}(m)\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x/q + 2\pi i m n/q} \Rightarrow$$

$$\Rightarrow \vartheta(x,\chi)\tau(\overline{\chi}) = \sum_{m=1}^{q}\overline{\chi}(m)\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x/q + 2\pi i m n/q} =$$

((*) con $x \rightsquigarrow x/q$ e $\alpha = m/q$)

$$= \sqrt{\frac{q}{x}}\sum_{m=1}^{q}\overline{\chi}(m)\sum_{n\in\mathbb{Z}} e^{-\pi(n+\frac{m}{q})^2 q/x} =$$

$$= \sqrt{\frac{q}{x}}\sum_{m=1}^{q}\overline{\chi}(m)\sum_{n\in\mathbb{Z}} e^{-\frac{\pi(nq+m)^2}{xq}} \overset{\overline{\chi}(m)=\overline{\chi}(nq+m)}{=}$$

$$= \sqrt{\frac{q}{x}}\sum_{\ell\in\mathbb{Z}}\overline{\chi}(\ell) e^{-\frac{\pi \ell^2}{xq}} = \sqrt{\frac{q}{x}}\vartheta\left(\frac{1}{x},\overline{\chi}\right) \Rightarrow$$

$$\Rightarrow \xi(s,\chi) = \frac{1}{2}\int_1^{+\infty}\vartheta(x,\chi)x^{s/2}\frac{dx}{x} + \frac{1}{2}\int_1^{+\infty}\underbrace{\vartheta\left(\frac{1}{x},\chi\right)}_{=\frac{1}{\tau(\overline{\chi})}\sqrt{qx}\,\vartheta(x,\overline{\chi})} x^{-\frac{s}{2}}\frac{dx}{x} =$$

($\frac{1}{x}=y$ nel pezzo da 0 a 1 e uso comunque x)

$$= \frac{1}{2}\int_1^{+\infty}\vartheta(x,\chi)x^{\frac{s}{2}}\frac{dx}{x} + \frac{1}{2}\cdot\frac{\sqrt{q}}{\tau(\overline{\chi})}\int_1^{+\infty}\vartheta(x,\overline{\chi})x^{\frac{1-s}{2}}\frac{dx}{x} \Rightarrow$$

$$\overset{s\rightsquigarrow 1-s,\ \chi\rightsquigarrow\overline{\chi}}{\Longrightarrow} \xi(1-s,\overline{\chi}) = \frac{1}{2}\int_1^{+\infty}\vartheta(x,\overline{\chi})x^{\frac{1-s}{2}}\frac{dx}{x} + \frac{1}{2}\frac{\sqrt{q}}{\tau(\chi)}\int_1^{+\infty}\vartheta(x,\chi)x^{\frac{s}{2}}\frac{dx}{x} =$$

$$= \frac{\sqrt{q}}{\tau(\chi)}\xi(s,\chi).$$

$$\frac{\sqrt{q}}{\tau(\overline{\chi})}\cdot\frac{\sqrt{q}}{\tau(\chi)} = \frac{q}{\tau(\overline{\chi})\tau(\chi)} \overset{\chi(-1)=1}{=} \frac{q}{|\tau(\chi)|^2} = 1$$

In questo caso, $L(s,\chi)$ ha zeri banali nei pari negativi e in 0. Nell'altro caso sono i dispari negativi.

L'altro caso: in maniera simile,

$$\left(\frac{\pi}{q}\right)^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)L(s,\chi) = \int_0^{+\infty}\sum_{n=0}^{+\infty} n\chi(n)e^{-\pi n^2 x}x^{\frac{s}{2}}\frac{dx}{x} =$$

$$= \frac{1}{2}\int_1^{+\infty}\vartheta_1(x,\chi)x^{\frac{s+1}{2}}\frac{dx}{x} + \frac{1}{2}\int_1^{+\infty}\vartheta_1\left(\frac{1}{x},\chi\right)x^{-\frac{s+1}{2}}\frac{dx}{x}.$$ Si trova, con (**),

$\vartheta_1(x,\chi) = \sum_{n\in\mathbb{Z}} n\chi(n)e^{-\pi n^2 x}$ $\qquad \vartheta_1(x,\chi)\tau(\overline{\chi}) = \dfrac{i\sqrt{q}}{x^{3/2}}\vartheta_1\left(\dfrac{1}{x},\overline{\chi}\right)$ e

si ottiene $\xi(1-s,\overline{\chi}) = \dfrac{i\sqrt{q}}{\tau(\chi)}\xi(s,\chi)$ (si usa $\tau(\overline{\chi}) = -\overline{\tau(\chi)}$ e $\chi(-n)=-\chi(n)$). □

Per $\sigma \geq 1/2$, $\Gamma(s+1/2) \ll e^{c_0|s|\log|s|} \ll_\varepsilon e^{|s|^{1+\varepsilon}}$ $\forall \varepsilon>0$

Per $\sigma>1$, $|L(s,\chi)| \ll \zeta(\sigma)$.

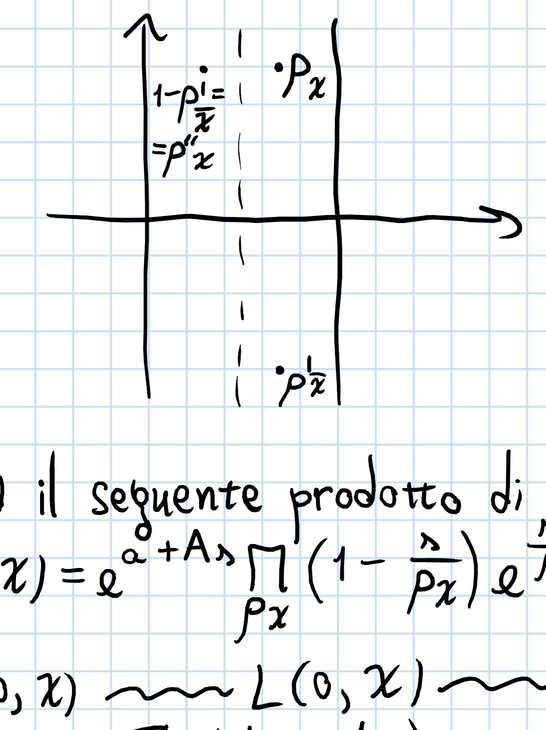
$$\sum_{n\leq x}\frac{\chi(n)}{n^s} = \underbrace{\left(\sum_{n\leq x}\chi(n)\right)}_{O(q)}x^{-s} + s\int_1^x\underbrace{\left(\sum_{n\leq u}\chi(n)\right)}_{O(q)}u^{-(s+1)}du \Rightarrow$$

$$\Rightarrow L(s,\chi) \overset{\sigma\geq\varepsilon}{=} s\int_1^{+\infty}\left(\sum_{n\leq u}\chi(n)\right)u^{-(s+1)}du \ll 2|s|q. \quad (\hookrightarrow \sigma\geq 1/2)$$

$\exists$ infiniti zeri $\rho_\chi = \beta_\chi + i\gamma_\chi$ con $0\leq\beta_\chi\leq 1$.

$$\sum_{\rho_\chi}\frac{1}{|\rho_\chi|^{1+\varepsilon}} < +\infty\ \forall\varepsilon>0 \quad \text{e} \quad \sum_{\rho_\chi}\frac{1}{|\rho_\chi|} = +\infty.$$

Non ci sono le stesse simmetrie: $L(\overline{s},\chi) = \overline{L(s,\overline{\chi})} \neq \overline{L(s,\chi)}$.



Prop.: si ha il seguente prodotto di Weierstrass:

$$\xi(s,\chi) = e^{a+As}\prod_{\rho_\chi}\left(1-\frac{s}{\rho_\chi}\right)e^{\frac{s}{\rho_\chi}}.$$

$$e^a = \xi(0,\chi) \sim\sim L(0,\chi) \sim\sim L(1,\chi) \ll \log q.$$

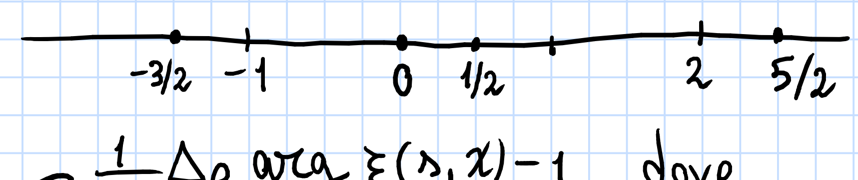
$$\operatorname{Re} A = -\frac{1}{2}\sum_{\rho_\chi}\left(\frac{1}{\rho_\chi}+\frac{1}{\overline{\rho_\chi}}\right) = -\sum_{\rho_\chi}\operatorname{Re}\left(\frac{1}{\rho_\chi}\right).$$

Cor.: si ha $\dfrac{\xi'}{\xi}(s,\chi) = A + \sum_{\rho_\chi}\left(\dfrac{1}{s-\rho_\chi}+\dfrac{1}{\rho_\chi}\right)$. $\left(s \in \underset{\text{cpt}}{K} \subset \mathbb{C}\setminus\bigcup\{\rho_\chi\}\right)$

$$\frac{L'}{L}(s,\chi) = \frac{1}{2}\log\left(\frac{\pi}{q}\right) - \frac{1}{2}\cdot\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) + A + \sum_{\rho_\chi}\left(\frac{1}{s-\rho_\chi}+\frac{1}{\rho_\chi}\right).$$

Prop. (formula di Riemann-Von Mangoldt): sia χ primitivo mod q e $T\geq 2$. Posto $N(T,\chi) = \frac{1}{2}\#\{\rho_\chi = \beta_\chi + i\gamma_\chi \mid 0\leq\beta_\chi\leq 1, |\gamma_\chi| < T\}$, si ha

$$N(T,\chi) = \frac{T}{2\pi}\log\left(\frac{qT}{2\pi}\right) - \frac{T}{2\pi} + O(\log(qT)).$$



$\frac{1}{2\pi}\Delta_R \arg\xi(s,\chi) - 1$ dove

$$R = \left\{\frac{5}{2}-iT, \frac{5}{2}+iT, -\frac{3}{2}+iT, -\frac{3}{2}-iT\right\}$$

$$\arg\xi(\sigma+it,\chi) = c + \arg\overline{\xi(1-\sigma+it,\chi)}$$

diventerà $2\cdot\frac{1}{2\pi}\Delta_L$, dove L è la metà destra di R.