$$\arg \xi(\sigma+it,\chi) = \arg \xi(1-\sigma-it,\bar\chi)+c = \arg \overline{\xi(1-\sigma+it,\chi)}+c$$

$$\xi(s,\chi) = \underbrace{\left(\frac{\pi}{q}\right)^{-\frac{s+a}{2}}\Gamma\left(\frac{s+a}{2}\right)}L(s,\chi)$$

χ primitivo mod q, $T\geq 2$, $N(T,\chi)=\frac{T}{2\pi}\log\left(\frac{qT}{2\pi}\right)+O(\cdots)$.

C'è da vedere il termine $\frac{1}{2\pi}\arg L(\frac12 \pm iT,\chi)$.

$$\frac{L'}{L}(s,\chi) = -\frac12\log\left(\frac{q}{\pi}\right)+A_\chi-\frac12\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right)+\sum_{\rho_\chi}\left(\frac{1}{s-\rho_\chi}+\frac{1}{\rho_\chi}\right)$$

$$\xi(s,\chi)=e^{a+A_\chi s}\prod_{\rho_\chi}\left(1-\frac{s}{\rho_\chi}\right)e^{s/\rho_\chi} \Rightarrow$$

$$\Rightarrow \frac{\xi'}{\xi}(s,\chi)=A_\chi+\sum_{\rho_\chi}\left(\frac{1}{s-\rho_\chi}+\frac{1}{\rho_\chi}\right)\Rightarrow$$

$$\Rightarrow \frac{\xi'}{\xi}(0,\chi)=A_\chi \qquad (A_{\bar\chi}=\bar A_\chi)$$

$\| \to$ eq. funzionale

$$-\frac{\xi'}{\xi}(1,\bar\chi) = -A_{\bar\chi}-\sum_{\rho_{\bar\chi}}\left(\frac{1}{1-\rho_{\bar\chi}}+\frac{1}{\rho_{\bar\chi}}\right)=$$

$$=-A_{\bar\chi}-\sum_{\rho_\chi}\left(\frac{1}{1-\bar\rho_\chi}+\frac{1}{\bar\rho_\chi}\right) = -A_{\bar\chi}-\sum_{\rho_\chi}\left(\frac{1}{\rho_\chi}+\frac{1}{\bar\rho_\chi}\right)\Rightarrow$$

$$\Rightarrow \operatorname{Re} A_\chi = -\sum_{\rho_\chi}\operatorname{Re}\left(\frac{1}{\rho_\chi}\right)$$

$$-\operatorname{Re}\frac{L'}{L}(s,\chi)=\frac12\log\left(\frac{q}{\pi}\right)-\cancel{\operatorname{Re}A_\chi}+\frac12\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right)-\sum_{\rho_\chi}\left(\operatorname{Re}\left(\frac{1}{s-\rho_\chi}\right)+\cancel{\operatorname{Re}\left(\frac{1}{\rho_\chi}\right)}\right)$$

$$-\operatorname{Re}\frac{L'}{L}(s,\chi)=\frac12\log\left(\frac{q}{\pi}\right)+\frac12\operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right)-\sum_{\rho_\chi}\operatorname{Re}\left(\frac{1}{s-\rho_\chi}\right).$$

$s=\sigma+it$, $-\operatorname{Re}\frac{L'}{L}(\sigma+it,\chi)\leq C\cdot\log(q(|t|+2))-\sum_{\rho_\chi}\frac{\sigma-\beta_\chi}{(\sigma-\beta_\chi)^2+(t-\gamma_\chi)^2}\Rightarrow$

$$\Rightarrow \sum_{\gamma_\chi}\frac{1}{1+(t-\gamma_\chi)^2}\ll\log(q\cdot(|t|+2))\Rightarrow \qquad \left(\sum_{|\gamma_\chi-t|<1}1\ll\log(q\cdot(|t|+2))\right)$$

$$\Rightarrow \frac{L'}{L}(s,\chi)=\sum_{\substack{\rho_\chi\\|\gamma_\chi-t|<1}}\frac{1}{s-\rho_\chi}+O(\log(q\cdot(|t|+2))).$$

$$\int_{1/2}^{1}\operatorname{Im}\frac{L'}{L}(\sigma\pm iT,\chi)\,d\sigma \ll \log(q\cdot(|T|+2))$$

$\|$

$$\arg\frac{L'}{L}\left(\frac12\pm iT,\chi\right)+O(1)$$

$$N(T,\chi)=\frac{T}{2\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{2\pi}+O(\log(qT)).$$

Oss.: se χ non è primitivo mod q, sia χ_1 mod q_1 che induce χ, allora $L(s,\chi)=L(s,\chi_1)\prod_{p|q}\left(1-\frac{\chi_1(p)}{p^s}\right)$ in $\mathbb{C}$, da cui

$\sigma\geq 1/2$ $$\frac{L'}{L}(s,\chi)=\frac{L'}{L}(s,\chi_1)+\sum_{p|q}\frac{p^{-s}\log p\,\chi_1(p)}{1-\frac{\chi_1(p)}{p^s}}\Rightarrow$$

$$\Rightarrow\left|\frac{L'}{L}(s,\chi)-\frac{L'}{L}(s,\chi_1)\right|\leq\sum_{p|q}\frac{\log p}{p^\sigma-1}\ll\sum_{p|q}\log p\leq\log q.$$

Ci sono gli zeri $0+it$ t.c. $p^{-it}=\pm1$, cioè $t=\frac{\pi(2k+1)}{\log p}$, che fino a T si stimano con $T\log p\Rightarrow\sum_{p|q}T\log p\leq T\log q$.

Allora $N(T,\chi)=\frac{T}{2\pi}\log T+O(T\log q)$.

Regioni libere da zeri per funzioni L formate con caratteri χ complessi

Prop.: se χ è complesso, allora $\exists$ una costante $c_0>0$ t.c.

$$\beta_\chi<1-\frac{c_0}{\log(q\cdot(|\gamma_\chi|+2))}=1-\frac{c_0}{\mathcal{L}}.$$

Dim.: $-\sum_{n=1}^{+\infty}\frac{\Lambda(n)\chi(n)}{n^{\sigma+it}}=\frac{L'}{L}(\sigma+it,\chi)\quad(\sigma>1)$

ricordiamo $3+4\cos\theta+\cos(2\theta)\geq0$

$$-\sum_{n=1}^{+\infty}\frac{\Lambda(n)\chi_0(n)}{n^\sigma}=\frac{L'}{L}(\sigma,\chi_0)$$

$-\sum_{n=1}^{+\infty}\frac{\Lambda(n)\chi^2(n)}{n^{\sigma+2it}}=\frac{L'}{L}(\sigma+2it,\chi^2)$. Se χ è complesso, posso fare come per la ζ. Se χ è reale no.

Viene $\frac{4}{\sigma-\beta_\chi}<\frac{3}{\sigma-1}+O(\log(q\cdot(|\gamma_\chi|+2)))$ con $\sigma=1+\frac{\delta}{\mathcal L}$.

$\hookrightarrow$ come per la ζ. □

Vediamo ora il caso χ reale.

$$\frac{L'}{L}(s,\chi^2)=\frac{L'}{L}(s,\chi_0)$$

$$\left|\frac{L'}{L}(s,\chi_0)-\frac{\zeta'}{\zeta}(s)\right|\leq\log q$$

$$-\operatorname{Re}\frac{\zeta'}{\zeta}(s)\ll\operatorname{Re}\frac{1}{s-1}+O(\log(|t|+2))$$

$$-\operatorname{Re}\frac{L'}{L}(\sigma+2i\gamma,\chi_0)$$
$$\rightsquigarrow -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+2i\gamma)\leq$$
$$\leq\operatorname{Re}\frac{1}{\sigma-1+2i\gamma}+c\mathcal L$$

$$\frac{4}{\sigma-\beta_\chi}<\frac{3}{\sigma-1}+\operatorname{Re}\left(\frac{1}{\sigma-1+2i\gamma_\chi}\right)+c\mathcal L$$

Supponiamo che $|\gamma_\chi|\geq\frac{\delta}{\log q}$; allora, prendendo $\sigma=1+\frac{\delta}{\mathcal L}$,

$$\frac{4\mathcal L}{\delta+(1-\beta_\chi)\mathcal L}<\frac{3\mathcal L}{\delta}+\frac{\delta\mathcal L}{5\delta^2}+c\mathcal L\Rightarrow$$

$$\Rightarrow 1-\beta_\chi>\frac{4-5c\delta}{16+5c\delta}\cdot\frac{\delta}{\mathcal L}=\frac{c'}{\mathcal L}\Rightarrow \text{tesi}.$$

Prop.: se χ è reale mod q, $\exists\, c_0>0$ t.c.

$$\beta_\chi<1-\frac{c_0}{\mathcal L}\qquad\text{se}\quad|\gamma_\chi|\geq\frac{1}{\log q}.$$

Prop.: se χ è reale mod q e $\rho_\chi=\beta_\chi+i\gamma_\chi$ con $\gamma_\chi\leq\frac{1}{\log q}$ e $\beta_\chi>1-\frac{c}{\mathcal L}$, allora ρ_χ è reale ($\gamma_\chi=0$) e semplice. È unico.

Dim.: $-\frac{L'}{L}(\sigma,\chi)<c\log q-\sum_{\rho_\chi}\operatorname{Re}\left(\frac{1}{\sigma-\rho_\chi}\right)$.

Se fosse $L(\beta_0\pm i\gamma_0,\chi)=0\Rightarrow-\frac{L'}{L}(\sigma,\chi)<c\log q-\frac{2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2}$

$$-\frac{L'}{L}(\sigma,\chi)=\sum_{n=1}^{+\infty}\frac{\chi(n)\Lambda(n)}{n^\sigma}\geq-\sum_{n=1}^{+\infty}\frac{\Lambda(n)}{n^\sigma}=\frac{\zeta'}{\zeta}(\sigma)>$$
$$>-\frac{1}{\sigma-1}+c_1.$$

Concatenando le disuguaglianze,

$$-\frac{1}{\sigma-1}<c\log q-\frac{2(\sigma-\beta_0)}{(\sigma-\beta_0)^2+\gamma_0^2}.$$

Prendendo $\sigma=1+\frac{2\delta}{\log q}$, si ottiene un assurdo se $|\gamma_0|<\frac{\delta}{\log q}$.

Con $\gamma_0=0$ è pure più facile. □