$$b_\chi = \begin{cases} L'/L(0,\chi) \\ \lim_{s\to 0}\left(\frac{L'}{L}(s,\chi) - \frac{1}{s}\right) \end{cases}$$

$$b_\chi = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) + A_\chi$$

$$\frac{L'}{L}(2,\chi) = -\frac{1}{2}\log\left(\frac{q}{\pi}\right) + O(1) + \sum_{\rho_\chi}\left(\frac{1}{2-\rho_\chi} + \frac{1}{\rho_\chi}\right) + A_\chi \Rightarrow$$

$$\Rightarrow b_\chi = -\sum_{\rho_\chi}\left(\frac{1}{2-\rho_\chi} + \frac{1}{\rho_\chi}\right) + O(1)$$

$$|\gamma_\chi| \geq 1 \Rightarrow \left|\sum_{|\gamma_\chi|\geq 1}(\dots)\right| \leq \sum_{|\gamma_\chi|\geq 1}\frac{2}{|\rho_\chi||2-\rho_\chi|} \ll \sum_{|\gamma_\chi|\geq 1}\frac{1}{1+\gamma_\chi^2} \ll \log q$$

$$|\gamma_\chi| < 1 \Rightarrow \left|\sum_{|\gamma_\chi|<1}\frac{1}{2-\rho_\chi}\right| \ll \sum_{|\gamma_\chi|<1} 1 \ll \log q \Rightarrow$$

$$\Rightarrow b_\chi = -\sum_{|\gamma_\chi|<1}\frac{1}{\rho_\chi} + O(\log q).$$

$$\psi(x,\chi) = -\sum_{|\gamma_\chi|<T}{}' \frac{x^{\rho_\chi}}{\rho_\chi} + \sum_{|\gamma_\chi|<1}{}'\frac{1}{\rho_\chi} - \left(\frac{x^{\beta_1}}{\beta_1} - \frac{1}{\beta_1}\right) - \left(\frac{x^{1-\beta_1}}{1-\beta_1} - \frac{1}{1-\beta_1}\right) + O\left(\frac{x\log^2(qx)}{T}\right)$$

$x \in \mathbb{N}$, $T \leq x$

tranne zeri di Siegel

$\beta_1 > 1 - \frac{c_0}{\log q} > \frac{3}{4}$ con $0 < c_0 < 1/4$ $(q \geq 3 \Rightarrow \log q > 1)$.

$1-\beta_1 < 1/4$, $\beta_1 > 3/4 \Rightarrow \frac{1}{\beta_1} = O(1)$.

$$\frac{x^{1-\beta_1}-1}{1-\beta_1} \underset{0\leq\sigma_0\leq 1-\beta_1<1/4}{=} x^{\sigma_0}\log x \ll x^{1/4}\log x$$

$$\sum_{|\gamma_\chi|<1}{}'\frac{1}{\rho_\chi} \ll \sum_{|\gamma_\chi|<1}{}'\frac{1}{\beta_\chi} \ll \log^2 q$$

$\sum 1 \ll \log q$, $\beta_\chi \geq \frac{c}{\log q}$

Formula esplicita per $\psi(x,\chi)$, χ carattere primitivo modulo q:

$$\psi(x,\chi) \overset{(*)}{=} -\sum_{|\gamma_\chi|\leq T}{}'\frac{x^{\rho_\chi}}{\rho_\chi} - \frac{x^{\beta_1}}{\beta_1} + O\left(\frac{x\log^2(qx)}{T} + x^{1/4}\log x\right).$$

Oss.: se χ é non primitivo e χ_1 induce χ,

$$\left|\sum_{n\leq x}\Lambda(n)\chi(n) - \sum_{n\leq x}\Lambda(n)\chi_1(n)\right| \leq \sum_{\substack{n\leq x,\\ (n,q)>1}}\Lambda(n) \ll \log^2(qx).$$

Oss.: $(*)$ vale $\forall \chi$ modulo q non principale.

Cor.: $(a,q)=1 \Rightarrow \psi(x;q,a) = \sum_{\substack{n\leq x\\ n\equiv a(q)}}\Lambda(n) = \frac{1}{\phi(q)}\sum_{\chi \bmod q}\overline{\chi}(a)\psi(x,\chi) =$

$$= \frac{x}{\phi(q)} + \frac{1}{\phi(q)}\sum_{\chi\neq\chi_0}\overline{\chi}(a)\psi(x,\chi) + O\left(\frac{x\exp(-c\sqrt{\log x})}{\phi(q)}\right).$$

$$\left|\sum_{|\gamma_\chi|<T}{}'\frac{x^{\rho_\chi}}{\rho_\chi}\right| \leq \sum_{1\leq|\gamma_\chi|<T}\frac{x^{\beta_\chi}}{|\gamma_\chi|} + \sum_{|\gamma_\chi|<1}\frac{x^{\beta_\chi}}{|\rho_\chi|} = \Sigma_1 + \Sigma_2$$

Σ_1:

$$\sum_{1\leq|\gamma_\chi|<T}\frac{1}{|\gamma_\chi|} \underset{\text{somma parziale, R-VM}}{\ll} \log^2(qT) \ll \log^2(qx)$$

$$x^{\beta_\chi} \leq x^{1-\frac{c}{\log(qT)}} = x\exp\left(-\frac{c\log x}{\log(qT)}\right)$$

Σ_2: $\sum_{|\gamma_\chi|<1}\frac{1}{|\rho_\chi|} \ll \log^2 q$, $x^{\beta_\chi} \leq x\exp\left(-\frac{c\log x}{\log(qT)}\right)$

$q \leq \exp(C(\log x)^{1/2})$ e $T = \exp(C(\log x)^{1/2}) \Rightarrow$

$$\Rightarrow \sum_{|\gamma_\chi|<T}\frac{x^{\rho_\chi}}{\rho_\chi} \ll x\exp(-c_1\sqrt{\log x}) \qquad \left(c_1 = \frac{c}{2C}\right).$$

In conclusione si ha

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x\exp(-c_1\sqrt{\log x})) - \frac{\chi(a)}{\phi(q)}\cdot\frac{x^{\beta_1}}{\beta_1}.$$

perché é reale

$z = \exp(C(\log x)^{1/2})$, allora la stima é unif. in q per $q \leq z$ e contiene un termine con β_1 unico $\left(\text{e } \beta_1 > 1 - \frac{c_0}{\log q}\right)$.

Usando $\beta_1 \leq 1 - \frac{c'}{\sqrt{q_1}\log^2 q_1}$ (q_1 il modulo "speciale"),

$x^{\beta_1} \leq x\exp\left(-\frac{c'\log x}{\sqrt{q_1}\log^2 q_1}\right)$. Voglio $q_1\log^4 q_1 \leq \log x$ (tipo $q_1 \leq (\log x)^{1-\delta}$),

allora $\psi(x;q,a) = \frac{x}{\phi(q)} + O(x\exp(-c_1\sqrt{\log x}))$ unif. per $q \leq (\log x)^{1-\delta}$.

é l'unica formula effettiva in tutte le variabili

GRH ($\beta_\chi = 1/2$ $\forall \rho_\chi \forall \chi$ modulo q) $\Rightarrow$

$\Rightarrow \psi(x,\chi) = O(x^{1/2}\log^2 x)$ $(T = \sqrt{x}, q \leq \sqrt{x})$,

allora $\psi(x;q,a) = \frac{x}{\phi(q)} + O(x^{1/2}\log^2 x)$.

Prop. (Bombieri): posto $E(x,q,a) = \psi(x;q,a) - \frac{x}{\phi(q)}$, si ha: $\forall A>0\ \exists B>0$ t.c.

$$\sum_{q\leq\frac{\sqrt{x}}{(\log x)^B}}\max_{(a,q)=1}\max_{y\leq x}|E(y,q,a)| \ll \frac{x}{(\log x)^A}.$$

Usando GRH, $\sum_{q\leq\frac{\sqrt{x}}{(\log x)^B}}\max_{(a,q)=1}|E(x,q,a)| \ll \sqrt{x}\log^2 x\sum_{q\leq(\dots)}1 \ll \frac{x}{(\log x)^{B-2}}$.

Allora basterebbe $B = A+2$. Nella prop., $B = A+5$.

Prop.: sia $\varepsilon > 0$. $\exists c(\varepsilon) > 0$ t.c. $L(1,\chi) > c(\varepsilon)q^{-\varepsilon}$.

Cor.: $\beta_1 > c(\varepsilon)q^{-\varepsilon}\log^2 q \geq c(\varepsilon)q^{-\varepsilon/2}$.

Conseguenza: $x^{\beta_1} < x\exp\left(-\frac{c(\varepsilon)\log x}{q^\varepsilon}\right)$. Se vogliamo $q \leq (\log x)^N$, $N>0$, basta scegliere $\varepsilon = \frac{1}{2N} \Rightarrow q^\varepsilon = q^{\frac{1}{2N}} \leq (\log x)^{1/2}$.

Prop. (Siegel-Walfisz): $\forall N>0\ \exists c(N)$ t.c.

$$\psi(x;q,a) = \frac{x}{\phi(q)} + O(x\exp(-c(N)(\log x)^{1/2})) \text{ unif.}$$

per $q \leq (\log x)^N$ (ma $c(N)$ e la costante nell'O-grande non sono calcolabili).

Dim. (della prima prop.): sia $F(s) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)$ con χ_1 e χ_2 primitivi reali (modulo q_1 e q_2).

$\operatorname{Res}_{s=1} F(s) = \lambda = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2)$.

Lemma: $\frac{7}{8} < s < 1 \Rightarrow F(s) > \frac{1}{2} - \frac{c\lambda}{1-s}(q_1q_2)^{8(1-s)}$.

Dim.: $F(s) = \sum_{n=1}^{+\infty}\frac{a_n}{n^s}$ $(\sigma>1)$, inoltre $a_1 = 1$ e i fattori del prodotto di Eulero sono

$$\left(1-\frac{1}{p^s}\right)^{-1}\left(1-\frac{\chi_1(p)}{p^s}\right)^{-1}\left(1-\frac{\chi_2(p)}{p^s}\right)^{-1}\left(1-\frac{\chi_1\cdot\chi_2(p)}{p^s}\right)^{-1} \Rightarrow$$

$$\Rightarrow \log F(s) = \sum_p\sum_m\frac{1}{mp^{ms}}\underbrace{(1+\chi_1(p^m))(1+\chi_2(p^m))}_{\geq 0} \Rightarrow a_n \geq 0.$$

$F(s) = \sum_{m=0}^{+\infty}b_m(2-s)^m$,

$$b_m = \frac{(-1)^m}{m!}\cdot F^{(m)}(2) = \frac{1}{m!}\sum_{n=1}^{+\infty}\frac{a_n(\log n)^m}{n^2} \underset{m\geq 1}{\geq} 0,\quad b_0 \geq 1.$$

$$F(s) - \frac{\lambda}{s-1} = \quad \frac{1}{s-1} = \sum_{m=0}^{+\infty}(2-s)^m$$

$= \sum_{m=0}^{+\infty}(b_m - \lambda)(2-s)^m$ e vale per $|2-s| \leq 3/2$ $(s \geq 1/2)$.

Da $|L(s,\chi)| \ll q|s|$, tra $\frac{1}{2}$ e 1 abbiamo

$L(s,\chi_j) \ll q_j$, $L(s,\chi_1\chi_2) \ll q_1q_2$ per $\frac{7}{8} \leq s \leq 1$.

Abbiamo $|b_m - \lambda| \underset{\text{Cauchy}}{\overset{\cdot|2-s|^m}{\leq}} c\,q_1^2q_2^2\cdot\left(\frac{2}{3}\right)^m\left(\frac{9}{8}\right)^m = c(q_1q_2)^2\left(\frac{3}{4}\right)^m \Rightarrow$

$$\Rightarrow \sum_{m=M}^{+\infty}|b_m - \lambda|(2-s)^m \leq c(q_1q_2)^2\left(\frac{3}{4}\right)^M$$

$$F(s) - \frac{\lambda}{s-1} = \sum_{m=0}^{M-1} + \sum_{m=M}^{+\infty} \geq 1 - \lambda\sum_{m=0}^{M-1}(2-s)^m - c(q_1q_2)^2e^{-M/4}.$$

Scegliamo M t.c. $\frac{1}{2}e^{-1/4} < c(q_1q_2)^2e^{-M/4} < 1/2 \Rightarrow$

$\Rightarrow F(s) - \frac{\lambda}{s-1} > \frac{1}{2} - \lambda\frac{(2-s)^M - 1}{1-s}$

$M < 8\log(q_1q_2) + c$

$$(2-s)^M = e^{M\log(1+1-s)} \leq e^{M(1-s)} \leq c_0(q_1q_2)^{8(1-s)} \Rightarrow$$

$$\Rightarrow \quad F(s) > \frac{1}{2} - \frac{\lambda c_0(q_1q_2)^{8(1-s)}}{1-s}. \quad \square$$

Sia $\varepsilon > 0$ fissato. 1° caso: $\exists \chi_1$ modulo q_1 $L(\beta_1,\chi_1) = 0$ con $\beta_1 > 1 - \frac{\varepsilon}{16}$, allora $F(\beta_1) = 0$. 2° caso: $L(1,\chi) > 0$ $L(\sigma,\chi) > 0$, $\zeta(\sigma) < 0 \Rightarrow$

$\Rightarrow F(\beta_1) < 0$, β_1 scelto. Allora $\exists \beta_1$ t.c.

$1 - \frac{\varepsilon}{16} < \beta_1 < 1$ t.c. $F(\beta_1) \leq 0 \Rightarrow$

$\underset{\text{lemma}}{\Rightarrow}$ $\frac{c_0\lambda}{1-s}(q_1q_2)^{8(1-s)} > \frac{1}{2}$. Prendo χ_2 modulo q_2 $(q_2 > q_1) \Rightarrow$

$$\Rightarrow \lambda \leq c_1\log q_1\log(q_1q_2)L(1,\chi_2).$$

$$L(1,\chi_2) \geq \frac{(1-\beta_1)}{2c}(q_1q_2)^{-8(1-\beta_1)}(\log q_1\log(q_1q_2))^{-1} \geq$$

$$\geq C(\chi_1)q_2^{-\varepsilon/2}(\log q_2)^{-1} \geq C(\chi_1)q_2^{-\varepsilon/3}. \quad \square$$