The Structure Theorem for Finitely Generated Modules over PIDs

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A brief history of the theorem

The Structure Theorem for Finitely Generated Modules over Principal Ideal Domains (PIDs) constitutes a generalised extension of a more elementary theorem known as ‘the Structure Theorem for Finitely Generated Abelian Groups’.

Indeed, this generalisation arises from the recognition that an abelian group can be viewed as a $\mathbb{Z}$-module.

The Invariant Factor Decomposition

Theorem

Let R be a PID and M be a finitely generated module over R. Then there exists a unique[1] *finite sequence (d_i) such that $d_1 \mid d_2 \mid \cdots \mid d_n$ and that:*

$$M \cong R/(d_1) \times \cdots \times R/(d_n).$$

The elements of (p_i) are called *invariant factors*.

[1] The sequence is unique up to multiplication by a unit.

The Primary Decomposition

Corollary

Let R be a PID and M be a finitely generated module over R. Then there exist unique[2] *prime powers $p_1^{k_1}$, ..., $p_n^{k_n}$ such that:*

$$M \cong R/(p_1^{k_1}) \times \cdots \times R/(p_n^{k_n}).$$

This corollary follows by applying the Chinese Remainder Theorem to the Invariant Factor Decomposition. The prime powers $p_i^{k_i}$ are called *elementary divisors*.

[2] The sequence is still unique up to multiplication by a unit.

Sketch of Proof

The proof is divided into three main parts:

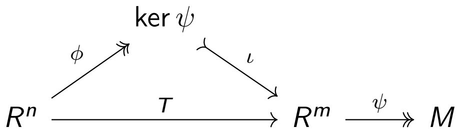
1. Reduction to a module homomorphism T from R^n to R^m,
2. Application of the Smith Normal Form to T,
3. Final application of the First Isomorphism Theorem.

Reduction to a module homomorphism T from R^n to R^m

1. Let m be the number of generators for M. Since M is finitely generated, there exists a surjective module homomorphism ψ from R^m to M. By the First Isomorphism Theorem, $R^m / \ker \psi \cong M$,
2. Since R^m is Noetherian, $\ker \psi$ is also finitely generated. Then there exists another surjective module homomorphism ϕ from R^n to $\ker \psi$, where n is the number of generators for $\ker \psi$,
3. Let ι be the natural inclusion from $\ker \psi$ into R^m. Then $T = \iota \circ \phi$ is a module homomorphism from R^n to R^m.
4. Since $\operatorname{im} T = \ker \psi$, we reduced the problem to understanding the module homomorphism T.

Reduction to a module homomorphism T from R^n to R^m

The following commutative diagram sums up the connections between the defined maps:



The following isomorphism holds:

$$M \cong R^n / \ker \psi \cong R^n / \operatorname{im} \phi \tag{1}$$

Application of the Smith Normal Form to T

Since T is a module homomorphism from the free-module R^n to the free-module R^m, T can be thought as a $m \times n$ matrix with elements in R.

Since R is a PID, there exist two bases for R^m and R^n that satisfy the Smith Normal Form for T. Therefore T has the following form in such bases[3]:

$$T' = \begin{pmatrix} d_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & d_k & \dots & 0 \end{pmatrix}$$

[3] The extra zeros might be under d_k if $m > n$.

Application of the Smith Normal Form to T

Let $\mathcal{B} = \{\underline{v_1}, \ldots, \underline{v_n}\}$ be a basis for R^n satisfying the Smith Normal Form for T. Then the following identity holds:

$$\operatorname{im} T = \langle d_1 \underline{v_1} \rangle \oplus \cdots \oplus \langle d_k \underline{v_k} \rangle \oplus \langle \underline{0} \rangle \oplus \cdots \langle \underline{0} \rangle.$$

Final application of the First Isomorphism Theorem

Let $\tau : R^n \to R/(d_1) \times \cdots \times R/(d_k) \times R/(0) \times \cdots R/(0)$ be a module homomorphism mapping $\alpha_1 \underline{v_1} + \ldots + \alpha_n \underline{v_n}$ to $(\overline{\alpha_1}, \ldots, \overline{\alpha_n})$. Its kernel is exactly im T. Therefore the First Isomorphism Theorem and the identity (1) imply the thesis:

$$M \cong R^n / \operatorname{im} T \cong R/(d_1) \times \cdots \times R/(d_k) \times R/(0) \times \cdots R/(0).$$

The uniqueness follows from the uniqueness of the Smith Normal Form.

Corollaries

Here is a list of the main corollaries of the Structure Theorem:

1. The Structure Theorem for Finitely Generated Abelian Groups,
2. The Jordan Normal Form (JNF),
3. The Frobenius Normal Form (also called the Rational Canonical Form),
4. All finite dimensional vector spaces are isomorphic to $\mathbb{K}^n$ for some $n \in \mathbb{N}$.